NONEXPANSIVE MAPPINGS AND FIXED POINTS

DISSERTATION

SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS
FOR THE AWARD OF THE DEGREE OF

Master of Philosophy

IN

MATHEMATICS

By

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2005

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CERTIFICATE

This is to certify that dissertation entitled ***"Non-expansive mappings and fixed points"*** has been completed by ***Miss Shazia Khurshid*** under my guidance. This work is more than adequate for the partial fulfillment for the award of the degree of Master of Philosophy in Mathematics.

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ACKNOWLEDGMENT

I bow in reverence to Almighty Allah, whose benign benediction gave me the required zeal for the completion of this work.

It is my immense pleasure to have the opportunity to express my deep sense of gratitude and indebtedness to my reverend supervisor, **Dr. Mohd. Imdad**, Reader, Department of Mathematics, Aligarh Muslim University, Aligarh, whose knowledgeable guidance and explicit enthusiasm helped me in several ways at various stages. His keen interest, sharp observations, careful and reasoned criticism remain a source of encouragement throughout the course of this study.

I must thank Prof. S. M. A. Zaidi, Chairman, Department of Mathematics, Aligarh Muslim University, Aligarh, for providing me the necessary facilities in the department. My acknowledgment goes to all the teachers of Department of Mathematics, Aligarh Muslim University, Aligarh, for extending their support.

Thanks are also due to all my seniors especially Dr. Ladlay Khan and Mr. Javid Ali, who had gone through the entire text and pointed out many omissions and errors beside suggesting several improvements. Their friendly advises and critical remarks deserve rare appreciation. I must appreciate the valuable support of some my colleagues Miss Gargi Varshney, Mr. Q. M. Danish Lohani and Miss Tarannum Kashmin.

I feel immensely delighted in placing on record greatest sense of gratitude to my personal friend Miss Effat Khanam for her unstinting support and encouragement, which infused confidence in me. The very rare rapport, I enjoy with her continues to be a great source of strength to me.

I am thankful to my sister Mrs. Farhana Syed and brother Mr. Aamir Khurshid whose elderly advises and spirited remarks have always acted as a moral booster especially in the course of doubt and despair.

I am indeed fortunate to be the daughter of two teachers that always get me into two institutions besides having very caring parent. I immensely owe much more to them than what I can possibly express in words. I dedicate this work to my parent whose love, support and encouragement used to be an epitome of strength to every individual.

I also accord my thanks to the seminar library staff Department of Mathematics, Aligarh Muslim University, Aligarh, for their cooperation.

07/05/06
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CONTENTS

PREFACE

In recent years, a great deal of work has been done in the field of nonlinear analysis. This topic has many interesting applications in various areas within and outside mathematics. 'Fixed Point Theory' is one of the most powerful tools to establish existence and uniqueness results in differential equations, integral equations, functional equations, partial differential equations and random differential equations. Besides this, fixed point theory has very fruitful applications in eigenvalue as well as boundary value problems which includes approximation theory, variational inequalities, complementarity problems and above all in mathematical economics. Obviously there is no clear line separating 'Fixed Point Theory' from other topological or set-theoretic branches as metric methods often inspire similar results in non-metric setting and vice-versa. Fixed points and fixed point theorems have always been a major theoretical tool in areas as widely apart as differential equations, topology, economics, game theory, dynamics, optimal control and functional analysis. Moreover, in recent years the applications of results and concepts from fixed point theory has increased enormously with the advent of accurate and efficient computational techniques enabling fixed point methods a major weapon in applied as well as pure mathematics.

The theory of nonexpansive mappings is closely related to the well known fixed point theory for contraction/contractive mappings which has been extensively studied for the past 40 years or so. The problem of classifying the family of Banach spaces for which every nonexpansive self mapping of a nonempty closed bounded and convex subset of Banach spaces has a fixed point continues to be a very hot subject of investigation for many good researchers of this domain which include the pioneer workers like Browder, Göhde and Kirk. In 1965, the following fundamental result was obtained: **If C is a closed bounded and convex subset of a uniformly convex space X and if $T : C \to C$ is nonexpansive then T has a fixed point**. In fact slightly variant forms of above result were obtained independently by Browder [19] Göhde [66] and Kirk [87]. Here it may be pointed out that Kirk [87] proved the above result for a more general form of set C which enjoys the so called normal structure property.

After the pioneer work of Browder [19], Göhde [66] and Kirk [87], extensive progress has been made in identifying what type of Banach space enjoy the fixed point or related properties. The problem of existence of a fixed point for a nonexpansive mapping is closely related to the structure of underlying Banach space. Sometimes the convexity assumption on C seems to be more suitable for fixed point theorems of some specific mappings (e.g. Schauder fixed point theorem uses completely continuous mappings) whereas the assumption like uniform convexity, reflexivity, normal structure, etc on the underlying space had usually been employed in linear functional analysis.

Inspired by the work of Browder [19], Göhde [66] and Kirk [87] a large number of fixed point results have been obtained for nonexpansive mappings using geometric properties of Banach spaces. Although a substantial number of definitive results have now been discovered, a few questions at the heart of the theory remain open and there is a lot of scope to address the extent to which the existing theory can be possibly extended. Some of these questions are merely tantalizing while others offer substantial new avenues of research. A long standing open problem was the following: **Does any Banach space X have the f.p.p.?**

The answer was given in 1981 by Alspach [2] who proved that $L^1[0,1]$ fails to have this property. The counter example due to Alspach [2] suggests the natural question to undertake: **Which Banach space do have f.p.p.?** No general answer is known to this question till date. In fact many special cases of this question also remain open, e.g.

Question 1. Does every reflexive space have f.p.p.?

Question 2. Does every Banach space which is isomorphic to a Hilbert space have f.p.p.?

Some classical fixed point theorems for single-valued nonexpansive mappings have been extended to multi-valued mappings but many questions remain open about the existence of fixed points for multi-valued nonexpansive mappings when the underlying Banach space enjoys specific geometric properties (e.g. X is nearly uniformly convex space).

Asymptotic fixed point theorems are those theorems from which the existence of fixed points of a mapping $T : X \to X$ are derived using the behavior of the iterates T^n(for some large n). In nonlinear analysis this theory has a long history but its extensive development in the metric sense is more recent. There are certain limitations of asymptotic theory as far as contraction mappings are concerned but there is a direction of asymptotic metric fixed point theory related to nonexpansive mappings which is nontrivial and presently receiving considerable attention. While proving results for asymptotically nonexpansive mapping one is generally motivated by the existing theory of nonexpansive mapping.

The main purpose of this work is to attempt an up to date but selective survey of nonexpansive and other related classes of mappings. The structure of this text is straightforward. There are four chapters devoted to various aspects of the theory. Each chapter is divided into various sections. Numbers like 1.2.4 indicate Subsection 4 of the Section 2 of the Chapter 1. The numbers in brackets refers to the references listed in the bibliography.

As usual Chapter 1 is devoted to the background material which begins with definitions such as convex sets, uniformly convex Banach spaces, strictly convex Banach spaces, locally uniformly convex Banach spaces, smooth Banach spaces, uniformly smooth Banach spaces, diametral point, normal structure, uniform normal structure, Opial's condition etc. Some classical fixed point theorems are also stated which includes: Brouwer fixed point theorem, Banach contraction principle, Schauder fixed point theorem, Caristi fixed point theorem and Tychonoff fixed point theorem besides discussing related results. The multi-valued analogue of these classical results are mentioned which also include some recent results of the existing literature. This chapter concludes with an introduction to various iteration procedures in fixed point theory which also includes some elementary but fundamental results via iterations.

Chapter 2 is dedicated to the classical theory of nonexpansive mappings which includes nice and natural results. This chapter begins with the pioneer work by Browder[19], Göhde [66] and Kirk [87]. In course of our survey we choose to cite the results due to Gossez and Dozo [68], Kannan [83], Penot [124], Kirk [93], Goebel and Koter [64] etc. In the last section, we have discussed some fixed point theorems via iteration which incorporate the work of Mann [111], Outlaw [122], Ishikawa [76, 77], Reich [132], Deng [39] and others.

In Chapter 3, we present fixed point theorems for multi-valued mappings in Banach spaces which begins with the work of Markin [112]. The allied concepts like weakly nonexpansive, *-nonexpansive and SL-contractive are also discussed and results involving such mappings are included which revolves around the work of Husain and Tarafdar [75], Husain and Latif [74], Xu [167], Benavides and Ramirez [8] and Shahzad and Lone [150]. This chapter concludes with some natural results regarding the structure of fixed point set for multi-valued mappings.

The main objective of Chapter 4 is to present some relatively recent fixed point theorems for asymptotically nonexpansive mappings. The first half of the chapter deals with the generalization of Browder-Göhde-Kirk theorem using certain iteration schemes which incorporate the work due to Schu [144, 145], Xu and Noor [164], Liu [109] etc. while the second half of the chapter looks at the asymptotically contractive and pseudocontractive mappings wherein we have a bird eye view on the results contained in Luc [110], Penot [125], Suzuki [155], Assad and Kirk [3] and Sharma and Sahu [151].

As usual, dissertation concludes with a bibliography which by no means is exhaustive one but lists only those books and papers which have been referred to in the dissertation.

LIST OF ABBREVIATIONS

$A = B, A \neq B$	: Equality and Inequality for sets
$[a,b], [a,b)$ e.t.c.	: Intervals on the real line
$\delta(A)$	: Diameter of a set A
$D(A,B)$	: Distance between the sets A and B
$D(x,A)$	: Distance between the point x and set A
$d(x,y)$	: Distance from one point to another
$\wedge$	: index set
inf	: Infimum (or greatest lower bond)
max	: Maximum
min	: Minimum
N	: Set of natural numbers
R	: Set of real numbers
R^+	: Set of non-negative real numbers
sup	: Supremum (or least upper bond)
$S\mathrm{o}T$	: S composition T
$\mid x \mid$	: Absolute value of x
$\|\cdot\|$	: Norm
$\Rightarrow$	: Implies
$\emptyset$	: Empty set
$\in$	: Belongs to, belonging to
$\notin$	: Does not belong to

CHAPTER 1

PRELIMINARIES

§ 1.1. Introduction

The term 'Metric Fixed Point Theory' refers to those theoretic results on fixed points in which geometric conditions on the underlying space and/or conditions on mapping play a crucial role. Many problems within mathematics have as their solutions the fixed point of some suitably realized mapping T and so a number of available procedures in numerical analysis and approximation theory facilitate to compute the fixed point of the required mapping via successive approximation. Fixed point theory is rich, interesting and has a wide range of applications. Fixed point theorems are very useful in the existence theory of differential equations, integral equations, functional equations, partial differential equations and random differential equations. Besides this, fixed point theory has very fruitful applications in eigenvalue as well as boundary value problems, including approximation theory, variational inequalities, complementarity problems and above all in mathematical economics.

The theory treated here has many contributors. Those who developed the classical theory include the celebrated mathematicians L. E. Brouwer, S. Banach, J. Schauder and A. Tychonoff. The fixed point theory received a new impetus in 1965 when fixed point theorems for nonexpansive mappings were discovered independently by Browder [19], Göhde [66] and Kirk [87] and along which related results were widely circulated in form of *Lecture Notes* due to Opial [121, 1967]. There have been numerous major discoveries since then. In this dissertation, we carry out an up to date but selected survey of fixed point results for nonexpansive and other related classes of mappings.

For a comprehensive study of fixed point theory and related results the books by Goebel and Kirk [63], Singh et al. [149], Dugundji and Granas [46] are of special recommendation.

The present chapter is to facilitate readability of the text, it consist of basic definitions and gives a brief survey of results on fixed point theory.

§ 1.2. Relevant definitions and results

One of the most important and interesting class of subsets of a linear space is the class of convex sets.

Definition 1.2.1. A subset C of a linear space X is said to be *convex* if $\alpha x + (1 - \alpha)y \in C$ whenever, $x, y \in C$ and $0 \leq \alpha \leq 1$.

A subset C of a linear space X is said to be *star shaped* if there is at least one $p \in C$ such that $(1-\alpha)p + \alpha x \in C$ for all $x \in C$ and $0 < \alpha < 1$. The point $p \in C$ is said to be the *star center* of C.

Notice that every convex set is star shaped but converse need not be true.

Definition 1.2.2. Let C be a subset of a Banach space X. Then the *convex hull* of C is defined as $c_0(C) = \{\Sigma \alpha_i x_i : 0 \le \alpha_i \le 1, \Sigma \alpha_i = 1, x_i \in C\}$, the set of all convex combination of points in C. For any subset C, $c_0(C)$ is the smallest convex subset of X containing C.

Definition 1.2.3. A Banach space X is called *uniformly convex* if for any $\epsilon > 0$ there exists a $\delta > 0$, depending on ϵ such that if $\|x\| = \|y\| = 1$ and $\|x - y\| \ge \epsilon$, then $\|\frac{1}{2}(x+y)\| \le 1 - \delta$.

In other words, X is uniformly convex if for any two points x and y on the unit sphere $S = \{x \in X : \|x\| = 1\}$, the midpoint of the line segment joining x and y can be close to but not on that sphere, only if x and y are sufficiently close to each other.

Example 1.2.1. Every Hilbert space and sequence space l^p, $1 < p < \infty$, are uniformly convex. However, $C[0,1]$ with sup norm, l^1 and l^∞ w.r.t. standard norms are not uniformly convex.

Definition 1.2.4. The *modulus of convexity* of a Banach space X is the function $\delta_X : [0,2] \to [0,1]$ defined by

$$\delta_X(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon\right\}.$$

A space X is uniformly convex if and only if its modulus of convexity satisfies $\delta_X(\epsilon) > 0$ for $\epsilon > 0$.

Definition 1.2.5. A Banach space X is called *strictly convex* if and only if $x, y \in X$, $x \ne 0 \ne y$ and $\|x+y\| = \|x\| + \|y\|$ imply that $x = \alpha y$, $\alpha > 0$. Equivalently, X is a strictly convex Banach space if whenever $x, y \in X$, $\|x\| = \|y\| = 1$ and $x \ne y$ then $\|\frac{1}{2}(x+y)\| < 1$.

A uniformly convex Banach space is strictly convex but not conversely. The spaces l^1 and L^1 are not strictly convex. Moreover, it is not difficult to see that the two concepts are equivalent in finite dimensional spaces (since balls in such spaces are compact).

The Banach space X^* of bounded linear functionals on X is called the dual space of X. It generates a topology for X called the *weak topology*. For given $\epsilon > 0$ and a finite number of elements $T_1, T_2, \ldots T_n$ in X^*, let

$$V(T_1, T_2, \ldots T_n; \epsilon) = \{x \in X : |T_i(x)| < \epsilon, \text{ for every } i = 1, 2, \ldots, n\}.$$

Then the family of all sets $V(T_1, T_2, \ldots, T_n; \epsilon)$ for every choice of ϵ and any finite sequence $\{T_1, T_2, \ldots, T_n\}$, defines a base of neighborhoods of zero of a topology which is called the *weak topology* of X. Under the weak topology, a normed linear space X is a locally convex topological vector space. In the sequel, by the terms weakly closed, weakly compact, weak closure of a set, we mean closed, compact, closure of a set with respect to the weak topology respectively.

The norm topology(or strong topology) and the weak topology of a Banach space X are equivalent if and only if X is finite dimensional. A sequence $\{x_n\} \subset X$ converges weakly to $y \in X$ if and only if $\lim_{n \to \infty} Tx_n = Ty$ for every $T \in X^*$.

We now collect some basic and well known properties about the weak topologies.

Theorem 1.2.1. Every weakly convergent sequence $\{x_n\}$ is bounded and moreover,

$$\| \lim_{n \to \infty} x_n \| \leq \lim_{n \to \infty} \inf \|x_n\|.$$

Theorem 1.2.2. Strong convergence implies weak convergence but not conversely.

Theorem 1.2.3. A convex subset C of a Banach space X is closed if and only if it is weakly closed.

Theorem 1.2.4. A weakly closed set is strongly closed but not conversely.

For every fixed vector $x \in X$, the mapping of X^* into $\Re$ or C, which to each $T \in X^*$ assigns the value Tx of T at x, is a continuous linear functional on X^*, i.e., an element of X^{**}. Moreover, the norm of this functional is equal to $\|x\|$. The canonical mapping of X into X^{**} defined by this correspondence between elements of X and X^{**} is linear and one to one. Therefore, it is an isometric embedding of X into X^{**}.

Definition 1.2.6. A Banach space X is called *reflexive* if the canonical embedding of X into X^{**} is onto.

Example 1.2.2. All Hilbert space and uniformly convex Banach space are reflexive. Also, sequence space l^p, $1 < p < \infty$ is reflexive.

Theorem 1.2.5. A Banach space X is reflexive if and only if one of the following (equivalent) condition holds:

(a) X^* is reflexive.

(b) $B(0,1)$ is weakly compact in X^*.

(c) Every bounded sequence in X admits a weakly convergent subsequence.

(d) For any $T \in X^*$ there exists a $x \in B(0,1)$ such that $Tx = \|T\|$.

(e) For any closed bounded and convex subset C of X and $T \in X^*$ there exists a $x \in C$ such that $Tx = \sup\{Ty : y \in C\}$.

(f) For any decreasing sequence $\{K_n\}$ of nonempty closed bounded and convex subsets of X, $\bigcap\limits_{n=1}^{\infty} K_n \neq \emptyset$.

A reflexive Banach space is not necessarily uniformly convex. For example, consider a finite dimensional Banach space in which the surface of the unit ball has a flat part. Such a Banach space is reflexive as its dimension is finite. But the flat portion in the surface of the ball forces it to be non uniformly convex.

There is another fact which distinguishes reflexive spaces from uniformly convex spaces. If $(X, \|.\|)$ is a Banach space and if $\|.\|_1$ is another norm on X which is equivalent to the norm $\|.\|$, then it readily follows that $(X, \|.\|)$ is reflexive if and only if $(X, \|.\|_1)$ is reflexive. Thus reflexivity is invariant under equivalent renormings. To see that this is not true of uniform convexity one need to look no further then the finite dimensional spaces $(\Re^n, \|.\|_2)$ and $(\Re^n, \|.\|_\infty)$.

Definition 1.2.7. Let $T : X \to Y$ be a mapping. Then T is said to be

(a) *demicontinuous* at x_0 if $x_n \to x_0 \Rightarrow Tx_n \to Tx_0$(weakly),

(b) *strongly continuous* at x_0 if $x_n \to x_0$(weakly) $\Rightarrow Tx_n \to Tx_0$,

(c) *weakly continuous* at x_0 if $x_n \to x_0$(weakly) $\Rightarrow Tx_n \to Tx_0$(weakly),

(d) *demiclosed* if $x_n \to x_0$(weakly) and $Tx_n \to y \Rightarrow y = Tx_0$.

Definition 1.2.8. Let C be a subset of a Banach space X. Then a mapping $T : C \to X^*$ is called *monotone* if

$$\langle Tx - Ty, x - y \rangle \geq 0, \quad \forall\, x, y \in C,$$

and *strictly monotone* if $\langle Tx - Ty, x - y \rangle > 0, \quad \forall\, x, y \in C \ (x \neq y)$.

Definition 1.2.9. A real-valued continuous function T defined on $\Re^+$ is called a *gauge function* if

(a) $T(0) = 0$,

(b) $\lim\limits_{t \to \infty} T(t) = +\infty$ and

(c) T is strictly increasing.

Definition 1.2.10. Let X be a Banach space and X^* its dual space. The *duality mapping* in X with gauge function T is a mapping $J : X \to 2^{X^*}$ such that $J(0) = 0$ and for $x \neq 0$,

$$J(x) = \{f \in X^* : f(x) = \|f\|\|x\|, \|f\| = T(\|x\|)\}.$$

An important example of a monotone mapping from a Banach space X into its dual space X^* is duality mapping.

Definition 1.2.11. A Banach space X is called *locally uniformly convex* (LUC) if and only if, for given $\epsilon > 0$ and an element $x_0 \in X$ with $\|x_0\| = 1$, there exists a $\delta(\epsilon, x_0) > 0$ such that

$$\left\|\frac{x_0 - y}{2}\right\| \leq 1 - \delta,$$

whenever $\|x_0 - y\| \geq \epsilon$ and $\|y\| = 1$.

It is clear from the definition that uniform convexity implies local uniform convexity but the converse is not true in general.

Definition 1.2.12. A Banach space X is *uniformly convex in every direction* (UCED) if and only if, for any $\epsilon > 0$ and every nonzero $z \in X$, there exists a number $\delta(\epsilon, z) > 0$ such that if $x - y = \alpha z$, $\|x\| = \|y\| = 1$ and

$$\left\|\frac{x + y}{2}\right\| > 1 - \delta, \text{ then } |\alpha| \leq \epsilon.$$

A uniformly convex space is a UCED but the converse is not always true. In fact, there are even reflexive Banach spaces that are UCED but not isomorphic to a uniformly convex Banach space.

The concept of normal structure plays a crucial role in some recent fixed point theorems for nonexpansive mappings in Banach spaces.

Definition 1.2.13. Let C be a nonempty bounded convex set in a Banach space X of diameter d. A point $x \in X$ is said to be a *diametral point* for C if

$$\sup_{y \in C} \|x - y\| = d.$$

Example 1.2.3. In the Banach space $C[0, 1]$, with $\|T\| = \max\limits_{0 \leq t \leq 1} |T(t)|$, every point of the bounded and convex set

$$C = \{T(t) \in C[0, 1] : 0 = T(0) \leq T(t) \leq T(1) = 1\}$$

is diametral.

Definition 1.2.14. A convex set C in a Banach space X is said to have *normal structure* if, for each bounded convex subset K of C, that contains more than one point, there exists a point $x \in K$ which is not diametral for K.

Geometrically, C has normal structure if, for every bounded and convex subset K of C, there exists a ball of radius less than the diameter of K centered at a point of K and containing K.

Remark 1.2.1. A Banach space is said to have a normal structure if each of its bounded convex subsets has normal structure.

Example 1.2.4.

(a) Every uniformly convex Banach space has normal structure.

(b) A Banach space, uniformly convex in every direction, has normal structure.

(c) Every convex and compact subset C of a Banach space has normal structure.

There are Banach spaces which do not possess normal structure. For example, the Banach spaces $C[0,1], l^1$ and L^1 do not have normal structure.

Definition 1.2.15. A convex subset C of a Banach space X is said to have *uniform normal structure* if there exists a constant $\alpha < 1$ such that any closed bounded and convex subset K of C for which $\delta(K) > 0$ contains a point for which

$$\sup\{\|x_0 - x\| : x \in H\} \leq \alpha.\delta(K).$$

Uniform normal structure is obviously stronger than normal structure and it mimics the normal structure behavior of uniformly convex spaces. Also, nonreflexive spaces exist which have normal structure. This contrasts with the following.

Theorem 1.2.6. A Banach space equipped with uniform normal structure is always reflexive.

Definition 1.2.16. A Banach space X is said to be *smooth* if for every $x \in X$ with $\|x\| = 1$, there exists a unique $T \in X^*$ such that $\|T\| = T(x) = 1$.

It is not difficult to prove that a Banach space X is smooth if and only if for every $x, y \in X$ with $x \neq 0$ the following limit exists:

$$\lim_{t \to 0} t^{-1}[\|x + ty\| - \|x\|] = \phi_x(y). \tag{1.2.1}$$

This limit defines a functional $\phi_x \in X^*$ which is called the *Gateaux derivative* of the norm at x.

Definition 1.2.17. A Banach space X is called *uniformly smooth* if the limit (1.2.1) exists uniformly in the set $\{(x, y) : \|x\| = \|y\| = 1\}$; thus X is uniformly smooth if for each $\epsilon > 0$ there exists $\delta > 0$ such that for $|t| < \delta$ and for all $x, y \in X$ with $\|x\| = \|y\| = 1$,

$$|\|x + ty\| - \|x\| - \phi_x(y)| < \epsilon|t|.$$

If the limit (1.2.1) exists uniformly for $\|y\| = 1$ when x is fixed, then the norm of X is said to be *Frèchet differentiable.* Spaces with Frèchet differentiable norm include all the classical spaces l^p, L^p, $1 < p < \infty$.

Definition 1.2.18. A Banach space X is said to satisfy *Opial's condition* if for each $x_0 \in X$ and each sequence $\{x_n\}$ in X weakly converging to x_0 the inequality

$$\liminf \|x_n - x\| > \liminf \|x_n - x_0\|$$

holds for all $x \neq x_0$.

L^p space $p \neq 2$ do not satisfy Opial's condition while every Hilbert space and $l^p(1 < p < \infty)$ space satisfy Opial's condition. Thus Opial's condition is independent of uniform convexity. On the other hand, Gossez and Lami Dozo [68] have observed that all such spaces have normal structure.

Theorem 1.2.7. If X is a reflexive Banach space which satisfies Opial's condition, then X has normal structure.

§ 1.3. Some basic fixed point theorems

In this section, we discuss some classical fixed point theorems, especially the Banach Contraction Principle [5] and some of its extensions. Though Banach's Contraction Principle [5] is remarkable in its simplicity, yet it is perhaps the most widely applied fixed point theorem in all of analysis.

Definition 1.3.1. Let T be a self mapping on a nonempty set X. A point $x \in X$ is called a *fixed point* of T if $Tx = x$, i.e., a point which remains invariant under the mapping T is called a fixed point of T.

Definition 1.3.2. A topological space X is called a *fixed point space* if every continuous mapping T of X into itself has a fixed point. The property of being a fixed point space is topologically invariant: for if X is a fixed point space and $T : X \to Y$ a homeomorphism, then for any $F : Y \to Y$ the map $T^{-1} \circ F \circ T : X \to X$ has a fixed point x_0, so $F \circ T(x_0) = T(x_0)$ and $T(x_0)$ is a fixed point for F.

Example 1.3.1. Any closed bounded interval $I = [a, b] \subset \Re$ is a fixed point space. Indeed, given $T : I \to I$, we have $a - T(a) \leq 0$ and $b - T(b) \geq 0$. Now applying the intermediate value theorem one concludes that the equation $x - T(x) = 0$ has a solution in I and therefore T has a fixed point.

Example 1.3.2. The Real line $\Re$ is not a fixed point space, as the translation $x \to x + \alpha$, $\alpha \neq 0$ has no fixed point.

In general, it is difficult to decide whether or not a given space is a fixed point space and usually results abstracting such situations have many interesting topological consequences. An example is the Brouwer fixed point theorem [15, 1912] which runs as follows:

Theorem 1.3.1. Let C be the closed unit ball in $\Re^n$ and $T : C \to C$ a continuous mapping. Then T has a fixed in C.

An immediate corollary of this theorem on the real line can be stated in the following way:

Corollary 1.3.1. Every continuous self mapping of a closed interval has a fixed point.

Definition 1.3.3. If X is a topological space and $C \subseteq X$, then a continuous mapping $r : X \to C$ is called a *retraction* if $r(x) = x$ for all $x \in C$. When this occurs, C is said to be a *retract* of X.

Theorem 1.3.2. Every closed convex subset C in $\Re^n$ is a retract of $\Re^n$.

The above fact permits an extension of Brouwer's Theorem to arbitrary closed bounded and convex subsets of $\Re^n$. For such a set C select a simplex S sufficiently large such that $C \subseteq S$. Thus there exists a retraction $r : S \to C$. The composition mapping $T \circ r$ is a continuous mapping of S into itself and therefore must have a fixed point, say x. Moreover, x must lie in the range of $T \circ r$, which is C. Since $r(x) = x$ for points in C,

$$x = T \circ r(x) = T(x).$$

Thus we have following.

Theorem 1.3.3. Let C be a closed bounded convex subset of $\Re^n$ and let $T : C \to C$ be continuous. Then T has a fixed point.

One very interesting consequence of Theorem 1.3.1 is the following fact.

Theorem 1.3.4. The surface S of a nontrivial closed ball B in $\Re^n$ is not a retract of B.

The following is the most useful formulation of Theorem 1.3.1.

Theorem 1.3.5. Suppose C is a nonempty closed bounded convex subset of a finite dimensional Banach space X and suppose $T : C \to C$ is continuous. Then T has at least one fixed point.

Most of the problems in functional analysis arise in sequence and function spaces which are mostly infinite dimensional, it is natural to ask whether Theorem 1.3.1 can be extended to infinite dimensional spaces. Kakutani [81] produced an example to show that Theorem 1.3.1 can not be extended to infinite dimensional spaces.

Example 1.3.3[81]. Let $C = \{x \in l^2 : \|x\| \leq 1\}$ be the unit ball in the Hilbert space l^2. With $x = \{x_1, x_2, x_3, ...\}$ in C, define a map $T : C \to C$ by

$$T(x) = \{\sqrt{1 - \|x\|^2}, x_1, x_2, ..., x_n, ...\}.$$

Then $\|Tx\| = 1$ and also T is continuous. But T admits no fixed point.

In 1930, Theorem 1.3.1 (due to Brouwer [15]) was extended to infinite dimensional spaces by Schauder [143] which is as follows:

Theorem 1.3.6[143]. Every continuous self mapping of a compact convex subset of a Banach space has at least one fixed point.

The compactness condition on subset is a stronger one. Many problems in analysis do not have a compact setting. It is natural to modify the theorem by relaxing the condition of compactness. Schauder [143] also proved a theorem for a compact mapping which is known as second form of Theorem 1.3.6. Before stating the theorem, we need the following definition.

Definition 1.3.4. A self mapping T of a Banach space X is called a *completely continuous compact mapping* if T is continuous and T maps bounded sets to precompact sets.

Remark 1.3.1. A compact mapping is always continuous but converse need not be true. For example, an identity mapping defined on an infinite dimensional normed space is continuous but not compact.

The following is another form of Theorem 1.3.6 due to Schauder [143].

Theorem 1.3.7[143]. Every compact self mapping of a closed bounded convex subset of a Banach space has at least one fixed point.

In 1935, Tychonoff [163] extended Brouwer's result to a compact convex subset of a locally convex topological vector space.

Theorem 1.3.8. A continuous self mapping of a nonempty compact convex subset of a locally convex topological vector space has a fixed point.

Definition 1.3.5. Let T be a self mapping of a metric space X. Then T is said to be of Lipschitz class if there exists a real number $k > 0$ such that $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in X$. If $k < 1$, then T is called contraction map. In case $d(Tx, Ty) < d(x, y)$, $x \neq y$, then T is said to be a contractive map.

The Banach Contraction Principle [5] is the simplest but one of the most versatile elementary results in the fixed point theory. The Banach Contraction Principle is stated below.

Theorem 1.3.9. Every contraction self mapping of a complete metric space has a unique fixed point.

Remark 1.3.2. All the conditions in Theorem 1.3.9 are necessary and cannot be relaxed.

The strength of the Banach Contraction Principle lies in the fact that the underlying space is quite general while the conclusion is very pointed. The fixed point is unique and the sequence of iterates of the original mapping for every point of the space always converge to the unique fixed point. Keeping in view the utility of Banach Contraction Principle, its several extensions and generalizations were attempted in recent years whose excellent survey is a available in Rhoades [138]. Here we opt to present some of its noted generalizations.

The first extension we take up which is due to Geraghty [55] was inspired by an earlier theorem of Rakotch [129].

Let $\mathcal{F}$ denote the class of those mappings $f : \Re^+ \to [0, 1)$ which satisfy the simple condition $f(t_n) \to 1 \Rightarrow t_n \to 0$.

Theorem 1.3.10[55]. Let T be a self mapping of a complete metric space (X, d) and suppose there exists $f \in \mathcal{F}$ such that for each $x, y \in X$,

$$d(Tx, Ty) \leq f(d(x, y))d(x, y).$$

Then T has a unique fixed point $z \in X$ and $\{T^n(x)\}$ converges to z for each $x \in X$

For the next result we suppose that S_1 denotes the collection of all monotone decreasing mappings $f : \Re^+ \to \Re^+$ for which $0 < f(t) < t$ and for which f is continuous from the right. This extension is due to Browder [19].

Theorem 1.3.11[19]. Let T be a continuous self mapping of a bounded complete metric space (X, d) and suppose there exists $f \in S_1$ such that for each $x, y \in X$

$$d(Tx, Ty) \leq f(d(x, y)).$$

Then T has a unique fixed point z and $\{T^n(x)\}$ converges to z for each $x \in X$.

Subsequently, Byod and Wong [13] obtained a more general result.

Theorem 1.3.12[13]. Let T be a self mapping of a complete metric space (X, d) which satisfies

$$d(Tx, Ty) \leq f(d(x, y)), \ \forall \ x, y \in X,$$

where $f : \Re^+ \to [0, \infty)$ is upper semi-continuous from the right and satisfies $0 \leq f(t) < t$ for $t > 0$. Then T has a unique fixed point z and $\{T^n(x)\}$ converges to z for each $x \in X$.

Following are also some recent extension of Banach Contraction Principle where neither T is continuous nor of contractive type.

Theorem 1.3.13[46]. Let T be a self mapping of a metric space X. Assume that for each $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that $d(x, Tx) < \delta$, implies $T(B_\epsilon(x)) \subset B_\epsilon(x)$. If $d(T^n y, T^{n+1} y) \to 0$ for some $y \in X$, then the sequence $T^n y$ converges to a fixed point of T.

Caristi [32] proved the following theorem where neither continuity nor a Lipschitz type condition is required. For this, we need the following definition.

Definition 1.3.6. Let X be a metric space. A function $T : X \to \Re$ is said to be lower(upper) semicontinuous at x_0 if

$$\liminf T(x) \geq T(x_0)(\limsup T(x) \leq T(x_0)) \text{ as } x \to x_0.$$

Theorem 1.3.14[32]. Let X be a complete metric space and $\phi : X \to [0, \infty)$ a lower semicontinuous function. If $T : X \to X$ is such that for each $x \in X$, $d(x, Tx) \leq \phi(x) - \phi(Tx)$, then T has a fixed point.

Notice that if we assume T is continuous, then the proof is a simple one. For any fixed $x_0 \in X$ let $x_n = T^n x_0$. Then

$$d(x_{n+1}, x_n) \leq \phi(x_n) - \phi(x_{n+1}).$$

Hence $\{\phi(x_n)\}$ is a decreasing sequence.
Now

$$\sum_{i=0}^{N} d(x_{i+1}, x_i) \leq \phi(x_0)$$

So $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, $\{x_n\}$ converges to $y \in X$. As T is continuous, $Ty = y$.

The proof of Theorem 1.3.14 without continuity of T is due to Takahashi [157] which is quiet technical and we opt not to present it's proof.

Remark 1.3.3.

(a) If $T : X \to X$ is a contraction map, i.e., $d(Tx, Ty) \leq kd(x, y)$, $0 \leq k < 1$, then condition of Theorem 1.3.14 is satisfied by taking $\phi(x) = \frac{1}{1-k} d(x, Tx)$.

(b) Let T be a self mapping of a metric space X, then there exists $\phi : X \to [0, \infty)$ satisfying $d(x, Tx) \leq \phi(x) - \phi(Tx)$ if and only if $\Sigma d(T^n x, T^{n+1} x)$ converges for all $x \in X$.

§ 1.4. Nonexpansive mappings

The class of contractive mappings can be enlarged as follows:

Definition 1.4.1. A self mapping T of a normed space X is said to be *nonexpansive* if for any $x, y \in X$,

$$\|Tx - Ty\| \leq \|x - y\|.$$

It is clear from the definition that a nonexpansive mapping is continuous. There exist nonexpansive mappings in which the fixed points abound.

Example 1.4.1. Let $X = \Re$ and $T : \Re \to \Re$ defined by

$$T(x) = x + \alpha, \quad \forall \ x \in \Re$$

which clearly has the property $d(Tx, Ty) = |x - y|$ and is without fixed points.

Example 1.4.2. Let $X = \Re \times \Re$ and define $T(x, y)$ by

$$T(x, y) = (x, 0).$$

Then T satisfies the above property and every point of the form $(x, 0)$ is a fixed point for this mapping, i.e., x-axis is invariant under T.

Remark 1.4.1. If T is a nonexpansive self mapping then any power of T (i.e., $T^n = TT \cdots T$) is again a nonexpansive mapping.

The following theorems in Banach spaces are fundamental in respect of nonexpansive mappings on arbitrary Banach spaces.

Theorem 1.4.1. Let C be a closed bounded and convex subset of a Banach space X and let $T : C \to C$ be nonexpansive. If $(I - T)C$ be a closed subset of X, then

T has a fixed point in C.

Theorem 1.4.2. Every nonexpansive self mapping of a compact convex subset C of a Banach space X has a fixed point.

Proof.

Take a fixed $x_0 \in C$ and define $T_{k_i} : C \to C$ by

$$T_{k_i}(x) = k_i Tx + (1 - k_i)x_0$$

where $0 < k_i < 1$ and $k_i \to 1$ as $i \to \infty$. Then each T_{k_i} is a contraction map and there is an x_{k_i} such that $T_{k_i} x_{k_i} = x_{k_i}$ by Banach contraction principle. The bounded sequence $\{x_{k_i}\}$ has a convergent subsequence $\{x_{k_{i_p}}\}$ which converges to $\overline{x}$, say.

We claim that $\overline{x}$ is a fixed point of T.

$$x_{k_{i_p}} = T_{k_{i_p}} x_{k_{i_p}} = (1 - k_{i_p})x_0 + k_{i_p} T_{k_{i_p}} x_{k_{i_p}}.$$

On letting $p \to \infty$, we get $\overline{x} = T\overline{x}$ since T is continuous and $k_{i_p} \to 1$.

The following celebrated theorem is due to Kirk [87].

Theorem 1.4.3. Let C be a closed bounded and convex subset of a reflexive Banach space X having normal structure. If $T : C \to C$ is nonexpansive, then T has a fixed point.

The following well-known result for nonexpansive mappings was proved independently by Browder [19], Göhde [66] and Kirk [87] which is popularly referred as Browder-Göhde-Kirk fixed point theorem.

Theorem 1.4.4. Let C be a nonempty closed bounded and convex subset of a uniformly convex Banach space X and $T : C \to C$ is nonexpansive, then T has a fixed point.

Definition 1.4.2. A self mapping T of a Banach space X is called *asymptotically regular* if $\lim\limits_{n\to\infty} \|T^{n+1}x - T^n x\| = 0$ for any $x \in X$.

Definition 1.4.3. Let X be a Banach space, $C \subseteq X$ and $T : C \to C$. Then T is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_i\} \subset \Re^+$ with $k_i \to 1$ as $i \to \infty$ such that

$$\|T^i(x) - T^i(y)\| \leq k_i \|x - y\| \quad \text{for all} \quad x, y \in C.$$

For asymptotically nonexpansive mappings it may be assumed that $k_i \geq 1$ and that $k_{i+1} \leq k_i$ for $i = 1, 2, ...$

Remark 1.4.2. A nonexpansive mapping need not be asymptotically regular, e.g., a translation mapping on a normed space is not asymptotically regular.

The following remarkable observations are given by DeFigueredo [40] regarding asymptotically regular mappings.

(a) If $\{x_n\}$ and $\{y_n\}$ are two sequences in a uniformly convex Banach space X such that $\|x_n\| \to 1$, $\|y_n\| \leq \|x_n\|$ and $\|\frac{x_n+y_n}{2}\| \to 1$ as $n \to \infty$, then $\|x_n - y_n\| \to 0$ as $n \to \infty$.
Indeed, let $z_n = \frac{x_n}{\|x_n\|}$ and $w_n = \frac{y_n}{\|x_n\|}$. Then $\|z_n\| = 1$, $\|w_n\| \leq 1$ and $\|\frac{z_n+w_n}{2}\| \to 1$. By uniform convexity, we get $\|z_n - w_n\| \to 0$, i.e.,$\|x_n - y_n\| \to 0$ as $n \to \infty$.

(b) The above observation (a) is also valid for $\lambda x_n + (1-\lambda)y_n$, $0 < \lambda < 1$, in place of $\frac{x_n+y_n}{2}$.

Browder and Petryshyn [29] proved the following fundamental result for a nonexpansive mapping defined on a Banach space.

Theorem 1.4.5. Let T be a nonexpansive self mapping of a uniformly convex Banach space X. If $F(T) \neq \emptyset$, then the mapping $T_\alpha = \alpha I + (1-\alpha)T$, $0 < \alpha < 1$ is nonexpansive and asymptotically regular. Moreover, $F(T) = F(T_\alpha)$.

Theorem 1.4.6. Let $T : X \to X$ be a nonexpansive asymptotically regular mapping in a Banach space X. Let $F(T)$ be nonempty and let T also satisfies the condition that $(I - T)$ maps closed bounded sets into closed sets. Then, for each $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to some point in $F(T)$.

Proof.
If $y \in F(T)$, then

$$\|T^{n+1}x_0 - y\| \leq \|T^n x_0 - y\|, \quad n = 1, 2, 3 \ldots$$

so the sequence $\{T^n x_0\}$ is bounded. Let G be the closure of $\{T^n x_0\}$. By given condition, it follow that $(I - T)G$ is closed. This, together with the fact that T is asymptotically regular, gives $0 \in (I - T)G$. So there exists some $z \in G$ such that $(I - T)z = 0$; yielding thereby $z = Tz$.

But this implies that either $z = T^n x_0$ for some n or there exists a subsequence $\{T^{n_i} x_0\}$ converging to z. Since z is a fixed point of T, it can be concluded that, in either case, the sequence $\{T^n x_0\}$ converges to z.

Remark 1.4.3. Let α be such that $0 < \alpha < 1$. Let $T_\alpha = \alpha I + (1-\alpha)T$. Then T satisfies given condition if and only if T_α does. To see this, note that

$$I - T_\alpha = (1-\alpha)(I - T).$$

In Hilbert space, the relationship between monotone and nonexpansive mappings is expressed by next result.

Theorem 1.4.7. Let C be a subset of a Hilbert space X and $T : C \to X$ a non-expansive mapping. Then the mapping $I - T$ is monotone (where I is the inclusion map).

Theorem 1.4.7 enables us to prove the following useful property of nonexpansive mappings in Hilbert spaces.

Theorem 1.4.8. For every nonexpansive mapping defined on a nonempty subset C of a Hilbert space X, the mapping $I - T$ is demiclosed.

Proof.

Let $\{x_n\} \subset C$ be a sequence that converges weakly to an element $x_0 \in C$ and a sequence $\{x_n - T(x_n)\}$ converges to an element y_0 in X. Then we have

$$\lim_{n\to\infty} \inf\|x_n - x_0\| \geq \lim_{n\to\infty} \inf\|Tx_n - Tx_0\| = \lim_{n\to\infty} \inf\|x_n - y_0 - Tx_0\|.$$

Note that, in a Hilbert space X, if $x_n \to x_0$ weakly and $x_0 \neq y$, then $\lim\limits_{n\to\infty} \|x_n - x_0\| < \lim\limits_{n\to\infty} \inf\|x_n - y\|$, so $x_0 = y_0 + Tx_0$. That is, $(I - T)x_0 = y_0$.

Theorem 1.4.9. Let X be a Hilbert space and C the closed ball $\{x \in X : \|x\| \leq r\}$. Then each nonexpansive mapping $T : C \to X$ has at least one of the following two properties:

(a) T has a fixed point,

(b) there exist $x \in \delta C$ and $\alpha \in (0, 1)$ such that $x = \alpha T(x)$.

Several fixed point theorems are obtained from Theorem 1.4.9 by imposing conditions that prevent occurrence of the second possibility.

Corollary 1.4.1. Let $C = \{x \in X : \|x\| \leq r\}$ and let $T : C \to X$ be nonexpansive. Assume that for all $x \in \delta C$, one of the following condition holds:

(a) $\|Tx\| \leq \|x\|$,

(b) $\|Tx\| \leq \|x - Tx\|$,

(c) $\|Tx\|^2 \leq \|x\|^2 + \|x - Tx\|^2$,

(d) $\langle x, Tx \rangle \leq \|x\|^2$,

(e) $Tx = -T(-x)$.

Then T has a fixed point.

In what follows, we show that some results on fixed point theorems can be obtained in the general setting of a Banach space, even when the hypothesis of non-expansiveness is considerably weakened. Indeed, the analysis involved in the proofs of various theorems on nonexpansive mappings does not require the full force of nonexpansiveness in case the existence of at least one fixed point is taken on hypothesis.

Definition 1.4.4. A mapping $T : X \to X$ with a nonempty fixed point set $F(T)$ is called *quasi-nonexpansive* if for $p \in F(T)$,

$$\|Tx - p\| \leq \|x - p\|$$

holds for all $x \in X$.

The concept of quasi-nonexpansive mapping was essentially introduced by Diaz and Metcalf [41] together with some other related ideas. But later Dotson [43] has labeled this concept as quasi-nonexpansive. It is clear that a nonexpansive mapping with at least one fixed point is quasi-nonexpansive. A linear quasi-nonexpansive mapping on a Banach space is nonexpansive. But there do exist continuous and discontinuous nonlinear mappings which are not nonexpansive. Dotson [43] gave the following example which is continuous quasi-nonexpansive but not nonexpansive.

Example 1.4.3. The mapping $T : \Re \to \Re$ defined by

$$Tx = \begin{cases} \frac{x}{2} sin\frac{1}{x}, & if\ x \neq 0 \\ 0, & if\ x = 0 \end{cases}$$

is quasi-nonexpansive but not nonexpansive.

Here, $Tx \neq x$ for any $x \neq 0$. Otherwise if $Tx = x$, then $x = \frac{x}{2}\sin\frac{1}{x}$ which is impossible. Also T is quasi-nonexpansive because for $x \in \Re$, we have

$$\|Tx - 0\| = \left\|\frac{x}{2}\right\| \left\|\sin\frac{1}{x}\right\| \leq \frac{\|x\|}{2} < \|x\| = \|x - 0\|$$

.

However, T is not nonexpansive map. Indeed, if $x_1 = \frac{2}{7\pi}$ and $x_2 = \frac{2}{17\pi}$, then $\mid x_1 - x_2 \mid= \frac{2}{\pi}(\frac{1}{7} - \frac{1}{17}) = \frac{20}{119\pi}$. Thus, $\mid Tx_1 - Tx_2 \mid>\mid x_1 - x_2 \mid$.

Dotson [43] proved the following result for quasi-nonexpansive mappings.

Theorem 1.4.10. Let C be a closed bounded and convex subset of a strictly convex Banach space X and let $T : C \to C$ be a quasi-nonexpansive mapping. Then $F(T)$ is a closed convex set and T is continuous on $F(T)$.

§ 1.5. Multi-valued mappings

This section deals with some basic observations about multi-valued mappings. In what follows, we use the following notations and definitions.

(a) 2^X = the family of all nonempty subsets of X.

(b) $CB(X) = \{C \in 2^X : C$ is closed and bounded$\}$.

(c) $K(X) = \{C \in 2^X : C$ is convex$\}$.

(d) $KK(X) = \{C \in 2^X : C$ is closed and convex$\}$.

(e) $\text{cpt}(X) = \{C \in 2^X : C$ is compact$\}$.

(f) $C(X) = \{C \in 2^X : C$ is compact and convex$\}$.

Definition 1.5.1. Let X and Y be two nonempty sets. A multi-valued mapping T from X to Y, denoted by $T : X \to Y$ is a subset of $X \times Y$. The *inverse of* T is the multi-valued mapping $T^{-1} : Y \to X$ defined by $(y, x) \in T^{-1}$ if and only if $(x, y) \in T$. The values of T are the sets $T(x) = \{y \in Y : (x, y) \in T\}$; *the fibers of* T are the sets $T^{-1}(y) = \{x \in X : (x, y) \in T\}$ for $y \in Y$. Thus, the value of T^{-1} for $y \in Y$ is the fiber $T^{-1}(y)$.

For $C \subset X$, the set

$$T(C) = \cup_{x \in C} T(x) = \{y \in Y : T^{-1}(y) \cap C \neq \emptyset\}$$

is called the *image of* C *under* T; for $K \subset Y$, the set

$$T^{-1}(K) = \cup_{y \in K} T^{-1}(y) = \{x \in X : T(x) \cap K \neq \emptyset\},$$

the image of K under T^{-1} is called the *inverse image of* K *under* T.

Definition 1.5.2. A multi-valued mapping $T : X \to X$ is said to have a fixed point if the point belong to its own image set (i.e., $z \in T(z)$ for some $z \in X$).

Definition 1.5.3. Let X and Y be topological spaces. A multi-valued mapping $T : X \to Y$ is called

(a) *upper semicontinuous* if the inverse image of a closed set is closed,

(b) *lower semicontinuous* if the inverse image of an open set is open.

A multifunction T is called continuous if it is upper as well as lower semicontinuous.

Example 1.5.1. If we define two multifunctions $F, T : \Re \to \Re$ as

$$Fx = \begin{cases} \{0\} & if\ x \neq 0 \\ [-1, 1], & if\ x = 0, \end{cases} \qquad Tx = \begin{cases} \{0\} & if\ x = 0 \\ [-1, 1], & if\ x \neq 0. \end{cases}$$

Clearly F is upper semicontinuous but not lower semicontinuous whereas T is lower semicontinuous but not upper semicontinuous.

For the relation between a continuous map and an upper semicontinuous map, we have the following:

Theorem 1.5.1. If $T : X \to \text{cpt}(X)$ is continuous, then it is upper semicontinuous.

Remark 1.5.1. The condition that the values of T are compact subsets is not removable in Theorem 1.5.1. As a matter of fact a nonexpansive mapping T on X into 2^X may fail to be upper semicontinuous.

Example 1.5.2[96]. Let $X = [0,1] \times [0,1] - \{(0,1)\}$ with the usual metric. Let $(x,y) \in X$, define

$$T(x,y) = \begin{cases} \text{the segment } \{(x,z) : z \in [0,1]\} \text{ if } x \neq 0 \\ \text{the segment } \{(0,z) : z \in [0,1)\} \text{ if } x = 0. \end{cases}$$

Then $T : X \to 2^X$ is nonexpansive on X but is not upper semicontinuous at $(0,y)$ for any $y \in [0,1)$. Because if we take

$$U = \{(x,y) \in X : x + y < 1\},$$

then U is open and contains $T(0,y)$. However U does not contain $T(x,z)$ for $(x,z) \in X$ and $x \neq 0$. Therefore no neighborhood of $(0,y)$ exists such that U contains the image of T at every point of the neighborhood. That is, T is not upper semicontinuous at $(0,y)$.

Theorem 1.5.2. If $T : X \to 2^X$ is upper semicontinuous, then the function F, where $F(x) = d(x, Tx)$ is lower semicontinuous.

Definition 1.5.4. Let X and Y be topological spaces. A multifunction $T : X \to Y$ is said to be

(a) closed if it is closed as a subset of $X \times Y$ and

(b) T is compact if the image $T(X)$ of X under T is contained in a compact subset of Y.

Following are some important and useful results in the theory of multi-valued mappings.

Theorem 1.5.3. Assume that X, Y and Z are topological spaces.

(a) If $T : X \to Y$ is upper semicontinuous with compact values and Y is Hausdorff, then T is closed.

(b) If $T : X \to Y$ is upper semicontinuous with compact values, then $T(C)$ is compact whenever $C \subset X$ is compact.

(c) If $T : X \to Y$ and $F : Y \to Z$ are upper semicontinuous, then $F \circ T$ is also upper semicontinuous.

Fixed point theory for multi-valued mappings was originally initiated by Neumann [119] in the study of game theory. In 1941, Kakutani [81] proved a generalization of Theorem 1.3.1 due to Brouwer [15] to multi-valued mappings.

Theorem 1.5.4[81]. Every upper semicontinuous multifunction T defined on a closed bounded and convex subset of $\Re^n$ with nonempty closed convex values has a fixed point.

The multi-valued analogue of Schauder's fixed point theorem was given by Bohnenblust and Karlin [10] as follows:

Theorem 1.5.5. Every upper semicontinuous multifunction defined on a nonempty compact convex subset of a Banach space with nonempty closed convex values has a fixed point.

The multi-valued analogue of Tychonoff's fixed point theorem was given by Fan [52] and Glicksberg [58] independently. They proved the following result.

Theorem 1.5.6. Every upper semicontinuous multi-valued mapping T defined on a nonempty compact convex subset of a locally convex Hausdorff topological vector space X with nonempty closed convex values admits a fixed point.

Let (X, d) be a metric space and $CB(X)$ denote the family of all nonempty closed bounded subsets of X. For $A, B \in CB(X)$

$$H(A,B) = \max\{\sup_{a\in A} d(a,B), \ \sup_{b\in B} d(b,A)\}.$$

H furnishes a metric on $CB(X)$ commonly called the Hausdorff-Pompeiu metric. It is elementary, yet technical to verify that completeness of X implies completeness of $(CB(X), H)$ (see, Blumenthal [9, 1953])

Definition 1.5.5. Let X be a metric space and $CB(X)$ the family of nonempty closed bounded subsets of X. A multifunction $T : X \to CB(X)$ is called a Lipschitz mapping with Lipschitz constant $k \geq 0$, if $H(Tx, Ty) \leq kd(x, y)$ for any $x, y \in X$. T is called nonexpansive if $k = 1$ and a set-valued contraction if $k < 1$.

Theorem 1.5.7. Let (X, d) be a metric space and $T : X \to CB(X)$ a Lipschitz mapping with Lipschitz constant k. If $x_n \to x_0$, then $d(x_n, T(x_n)) \to d(x_0, T(x_0))$;

that is, $d(x, T(x))$ is a continuous function of x.

Nadler [116] gave the following multi-valued analogue of Banach Contraction Principle.

Theorem 1.5.8. Every multi-valued contraction $T : X \to CB(X)$ defined on a complete metric space X has a fixed point.

Before closing this section, we give the notion of semiconvex mappings due to Ko [96] as follows:

Definition 1.5.6. Let C be a convex subset of a normed space X. The mapping $T : C \to CB(X)$ is said to be *semiconvex* on C, if for any $x, y \in C$, $z = \alpha x + (1-\alpha)y$ where $0 \leq \alpha \leq 1$ and for any $x_1 \in Tx$, $y_1 \in Ty$, there exists $z_1 \in Tz$ such that

$$\|z_1\| \leq \max \{\|x_1\|, \|y_1\|\}.$$

Remark 1.5.2. A convex mapping is semiconvex but the converse is not true. Consider the mapping $T(x) = \sqrt{x}$, $x \in [0,1]$, for instance. The mapping T is semiconvex because it is strictly increasing. But T is not convex because if we take $x = 1$ and $y = 0$,

$$z = 1/4 = 1/4 \cdot 1 + 3/4 \cdot 0,$$

then $T(1) = 1, T(0) = 0$, but

$$T(z) = \sqrt{1/4} = 1/2 \not\leq 1/4\, T(1) + 3/4\, T(0) = 1/4.$$

§ 1.6. The iteration process

In an iteration process, we choose an arbitrary point x_0 in a given set and calculate recursively a sequence $\{x_0, x_1, x_2, \ldots\}$ from a relation of the form

$$x_{n+1} = Tx_n,$$

i.e., for arbitrary x_0 one successively writes $x_1 = Tx_0$, $x_2 = Tx_1 = T^2x_0, \ldots$. With the development of fast computers, iteration schemes are used in nearly every branch of applied mathematics and convergence proofs and error estimates are very often obtained using some fixed point theorems.

We begin with the following results which deal with the convergence of the sequence of iterates for continuous functions defined on closed intervals. The following result is due to Hillam [70].

Theorem 1.6.1. If $T : [0,1] \to [0,1]$ is a continuous function and x_0 in $[0,1]$ is any arbitrary point, then the sequence of iterates given by $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots.$ converges to a fixed point of T if and only if $\lim |x_{n+1} - x_n| = 0$ as $n \to \infty$.

It is interesting to note that for a continuous function $T : [a,b] \to [a,b]$, the following statements are equivalent.

(a) T is asymptotically regular; that is, $\lim_{n\to\infty} |x_{n+1} - x_n| = 0$ for all x in $[a,b]$ where $x_{n+1} = Tx_n$.

(b) T admits no cycle of order 2; that is, if for each x in $[a,b]$ with $x \neq Tx$, then $x \neq T^2x$.

(c) $\{T^n(x)\}$ converges for each x in $[a,b]$.

For nonexpansive mapping, we have the following result due to Bailey [4].

Theorem 1.6.2. If $T : [a,b] \to [a,b]$ is a nonexpansive mapping, then $x_{n+1} = \frac{1}{2}\{Tx_n + x_n\}$ converges to a fixed point of T.

The following is due to Hillam [70].

Theorem 1.6.3. Let $T : [a,b] \to [a,b]$ be a mapping such that $|Tx - Ty| \leq k|x-y|$ for all $x, y \in [a,b]$. Let $x_1 \in [a,b]$ be arbitrary and $x_{n+1} = (1-\alpha)x_n + \alpha Tx_n$ where $\alpha = \frac{1}{1+k}$. Then $\{x_n\}$ converges monotonically to a fixed point of T.

The sequence of successive approximation for nonexpansive mapping, unlike contraction mapping, may fail to converge, e.g., rotation about the origin in a plane where $x_{n+1} = Tx_n (x_0 \neq 0)$ does not converge.

An early result, regarding the convergence of sequence of successive approximation is due to Krasnoselskii [85].

Theorem 1.6.4. Let X be a uniformly convex Banach space and C a closed bounded and convex subset of X. If $T : C \to C$ is nonexpansive and $\overline{T(C)}$ is compact, then the mapping defined by

$$T_{\frac{1}{2}}x = \frac{1}{2}x + \frac{1}{2}Tx$$

has the property that its sequence of iterates always converges to a fixed point of T.

Since T and $T_{\frac{1}{2}}$ have the same fixed points, the limit of a convergent sequence given by

$$x_{n+1} = \frac{1}{2}x_n + \frac{1}{2}Tx_n$$

is necessarily a fixed point of T.

More generally, if C is a convex set in a Banach space X and $T : C \to C$ is a nonexpansive mapping, then for $\alpha \in (0,1)$,

$$T_\alpha x = \alpha x + (1-\alpha)Tx$$

is a nonexpansive map and has the same fixed points as T. Schaefer [142] proved that the sequence $x_{n+1} = \alpha x_n + (1-\alpha)Tx_n$ converges to a fixed point of T under the assumptions of Theorem 1.6.4. Since a nonexpansive mapping may have more than one fixed point, the limit of $x_{n+1} = \alpha x_n + (1-\alpha)Tx_n$ can depend on x_0 and on α as well.

Edelstein [47, 1966] succeeded in relaxing the condition of uniform convexity and proved Theorem 1.6.4 for strictly convex Banach spaces. Diaz and Metcalf [41] gave the theorem for strictly Banach spaces for the sequence of the type

$$x_{n+1} = \alpha x_n + (1-\alpha)Tx_n.$$

In 1971, Petryshyn [126] extended the result to densifying nonexpansive mappings. Whose definition involve the notion of measure of noncompactness described as follows:

Definition 1.6.1. Let C be a bounded subset of a metric space X. Then one defines the *measure of noncompactness* $\alpha(C)$ of C by

$$\alpha(C) = \inf\{\epsilon > 0 : \ C \ \text{ admits a finite covering of subsets of diameter} \leq \epsilon\}.$$

Definition 1.6.2. Let $T : X \to X$ be a continuous mapping of a Banach space X. Then T is called a *k-set contraction* if, for all bounded $C \subset X$ $T(C)$ is bounded and $\alpha(TC) \leq k\alpha(C)$, $0 < k < 1$. If

$$\alpha(TC) < \alpha(C), \quad \text{for all } \alpha(C) > 0,$$

then, T is called *densifying (or condensing).*

A k-set contraction, with $k \in [0, 1)$ is densifying but the converse is not true.

If $\alpha(T(C)) \leq \alpha(C)$, then T is called a 1-set contraction.

Theorem 1.6.5. Let C be a closed bounded and convex subset of a strictly convex Banach space X and $T : C \to C$ a densifying nonexpansive mapping. Let $T_\alpha x = \alpha x + (1-\alpha)Tx$ for constant α, with $0 < \alpha < 1$. Then, for each $x_n \in C$, the sequence $x_{n+1} = \alpha x_n + (1-\alpha)Tx_n$, $n = 0, 1, 2, \ldots$ converges strongly to a fixed point of T in C.

The following theorem is due to Petryshyn and Williamson [128].

Theorem 1.6.6. Let C be a closed subset of a Banach space X and let $T : C \to X$ be continuous mapping such that

(*a*) $F(T) \neq \emptyset$,

(*b*) for each $x \in C$ and every $p \in F(T)$, $\|Tx - p\| \leq \|x - p\|$, and

(*c*) there exists an $x_0 \in C$ such that $x_n = T^n x_0 \in C$ for each $n \geq 1$.

Then $\{x_n\}$ converges to a fixed point of T in C if and only if $\lim_{n\to\infty} d(x_n, F(T)) = 0$.

In 1981, Das, Singh and Watson [34] gave the following result.

Theorem 1.6.7. Let C be a closed subset of a Banach space X and let $T : C \to X$ be a quasi-nonexpansive mapping. Suppose that $x_1 \in C$ is such that

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n$$

yields $\{x_n\}$ either as a sequence in a normal Mann process (Def. 2.3.1) or a sequence of iterates $(x_{n+1} = Tx_n)$. If $\lim_{n\to\infty} d(x_n, F(T)) = 0$, then $\{x_n\}$ converges to a fixed point of T.

Remark 1.6.1.

(*a*) In this case, $\sum \alpha_n$ being divergent is not required. Also, it is evident that for $\alpha_n = 1$ (for each n), this theorem holds for complete metric spaces.

(*b*) If $\alpha_n = 1$ for each n, one gets Theorem 1.6.3 of [128] as long as T is not assumed to be continuous.

In 1970, Dotson [42] considered the iteration process given below and discussed the convergence of the sequence given by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n.$$

Theorem 1.6.8. Let C be a closed subset of a Banach space X and let $T : C \to X$. Let $\{\alpha_n\}$ be a sequence such that $\alpha_n \in (0,1)$ for each n. Let $x_1 \in C$ be such that x_{n+1} is defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \ \forall\, n.$$

If $0 < \alpha_n < 1$ and $\sum \alpha_n$ diverges, then $\{x_n\}$ is a normal Mann process [111].

Results in the same direction were also given by Reinermann [135] and Rhoades [136].

Recently, Edelstein and O'Brien [49] and Ishikawa [77] independently proved that even strict convexity in Theorem 1.6.5 is not essential. Edelstein and O'Brien [49] considered

$$T_\alpha x = \alpha T x + (1 - \alpha)x, \ (T_\alpha^n(x_0) = x_n)$$

whereas Ishikawa considered the sequence of the type

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n,$$

where $\{\alpha_n\} \in (0,1), \alpha_n \leq l < 1$ and $\sum \alpha_n = +\infty$.

They proved independently that the sequence defined above has the property that $\lim \|x_{n+1} - x_n\| = 0$ as $n \to \infty$. If the range of T is precompact, then the sequences $\{T_\alpha^n x_0\}$ and $\{x_n\}$, where $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n$, converge to a fixed point of T.

The following is also due to Dotson [44].

Theorem 1.6.9. Let X be a Hilbert space and $f : X \to X$ a monotonic nonexpansive operator on X. For $y_0 \in X$, define $T : X \to X$ by $Tu = -fu + y_0$ for all $u \in X$. Suppose $0 \leq \alpha_n \leq 1$ for all $n = 1, 2, ...,$ and $\sum_{n\to\infty} \alpha_n(1-\alpha_n)$ diverges. Then

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n$$

converges to the unique solution of the equation $u + Tu = y_0$.

Hirano and Takahashi [71] proved the following for asymptotically nonexpansive mapping.

Theorem 1.6.10. Let C be a closed convex subset of a Hilbert space X and $T : C \to C$ be such that

(a) T is asymptotically nonexpansive, and

(b) for each $z \in C$, $\{T^n z\}$ is bounded.

Then for each $x \in C$, $s_n(x) = \frac{1}{n} \sum_{i=0}^{n-1} T^i x$ converges weakly to a fixed point of T.

* * * * *

CHAPTER 2

SELECTED FIXED POINT THEOREMS FOR NONEXPANSIVE MAPPINGS

§ 2.1. Introduction

As discussed in Chapter 1^{st}, a mapping T from a nonempty subset C of X, (where X is a metric space with metric d) is called *nonexpansive* if its Lipschitz constant k does not exceed 1. Thus this class of mappings includes the classes of contractions, strictly contractive mappings and isometries (including the identity). Explicitly speaking, $T : C \to X$ is nonexpansive if

$$d(Tx, Ty) \leq d(x, y), \quad \forall\, x, y \in X.$$

Banach Contraction Principle is one of the greatest theorems ever proved in mathematics. Its setting is very natural as the sequence of iterates for every point of a complete metric space converges due to contraction condition which turns out to be the unique fixed point of the involved map. Thus this principle also provide an excellent method of computation of fixed point in its specific setting using sophisticated method of computation. The theory of fixed point of nonexpansive mappings is relatively much more cumbersome and tricky as sequence of iterates of such mapping need not converge. The following example illustrate this fact better.

Example 2.1.1. Let $T : [-1, 1] \to [-1, 1]$ be given by $Tx = -x$. Then for $x_0 \neq 0$, $x_{n+1} = Tx_n$ does not converge to $0 = T(0)$.

Clearly T is a nonexpansive mapping and has a fixed point but the iterative procedure fails to converge.

Example 2.1.2. $T : \Re \to \Re$ is given by $T(x) = 1 - x$, then $x_{n+1} = Tx_n$ gives, for $x_0 = 1$ say, $x_{2n} = 1$ and $x_{2n+1} = 0$ for $n \geq 1$.

Nonexpansive mapping may be fixed point free (e.g. translation mapping) and obviously when such a mapping has a fixed point it need not be unique, e.g., the identity mapping. The first important result in the theory of fixed points for nonexpansive mappings was obtained by R. deMarr [38] who has proved an interesting extension of the famous results of Kakutani [80] and Markov [113].

Thus theory of fixed point of nonexpansive mappings form a natural subject for investigation. We begin with a natural question that on general Banach spaces nonexpansive mappings may fail to have fixed points (as demonstrated by Sadovskii [140]).

Example 2.1.3[140]. Let c_0 be the Banach space of all sequences converging to zero with the norm,

$$x = (x_i) \to \|x\| = \sup_{i \geq 1} |x_i|$$

and let C be the set of all $x \in c_0$ such that $\|x\| \leq 1$. Then C is a closed bounded and convex set in c_0. Define following map on C,

$$Tx = (1, x_1, x_2, x_3, \ldots)$$

where $x = (x_i)$. It is obvious that T is a nonexpansive mapping on C with values in C and, moreover, that T is an isometry, i.e., for any x, y in C,

$$\|Tx - Ty\| = \|x - y\|.$$

If T has a fixed point in C then this is necessarily of the form $(1, 1, 1, \ldots)$ which is not in c_0.

This example suggests that in order to obtain positive results in the problem of existence of fixed points for nonexpansive mappings it is necessary to impose some restriction either on the mapping or the space.

The problem of determining conditions on C (or on the space X containing C) ensuring the existence of fixed points of nonexpansive mappings defined on C has its origin in four papers which appeared in 1965. In the first of these Browder [17], proved that a closed bounded convex set $C \subset X$ has f.p.p. if X is a Hilbert space. Almost simultaneously, both Browder [19] and Göhde [66] proved that the same is true if X belongs to the much wider class of 'uniformly convex' space. At the same time Kirk [87] observed that the presence of a geometric property called 'normal structure'(see Def. 1.2.14) guarantees that $C \subset X$ has f.p.p. if X is reflexive. The concept of normal structure was introduced in 1948 by Brodskii and Milman [14] to study fixed points of isometries and it is a property shared by all uniformly convex spaces. This line of argument has been exploited to yield further results on the existence and calculation of fixed points of nonexpansive mappings in Hilbert spaces and in certain classes of Banach spaces, for one can be referred to Browder [16,18,20,21,23,24] Browder and Petryshyn [28, 29] and Opial [120].

§ 2.2. Nonexpansive mappings on some classes of Banach spaces

In this section, we present results obtained on the existence of fixed points of nonexpansive mappings when underlying Banach space enjoys some geometric properties such as uniform convexity, reflexivity or property of normal structure etc. Fixed point theorems for nonexpansive mappings can be divided into two main classes: the one class is consisted of those theorems which make heavy use of topological notion such as continuity, compactness etc.; important results in this area are given by Tychonoff [163], Lefschetz [101], Ky Fan [52] and many others. The other class contains results proved in metrical spaces using convexity and dealing with nonexpansive mappings: related names of beginners are, among others, those of Browder, Göhde and Kirk. In fact, much work concerning this second class of problems was done around 1965. In what follows, we are going to discuss results of the second type, which are related to geometry of Banach spaces.

The following theorem is given by Kirk [87, 1965] wherein he used the following characterization of reflexivity due to Smulian [153] and a concept of Brodskii and Milman [14] to prove a fixed point theorem for mappings which do not increase distances.

Theorem 2.2.1[153]. Necessary and sufficient condition for a Banach space X to be reflexive is that, every bounded descending sequence of nonempty closed convex subsets of X have a nonempty intersection.

Theorem 2.2.2[87]. Every nonexpansive self mapping of a nonempty closed bounded and convex subset of a reflexive Banach space X possessing normal structure has a fixed point.

Proof.

Let $\Phi = \{K \subset C : T(K) \subset K; K$ is nonempty closed and convex $\}$. Since $C \in \Phi$, Φ is nonempty; Φ can be partially ordered by set inclusion.

A chain ψ in Φ has the finite intersection property.

Now, as a closed bounded convex set in a reflexive Banach space, C is weakly compact. Therefore, the family ψ of weakly closed subset of Φ has nonempty intersection, i.e., $G = \cap_{C_i \in \psi} C_i \neq \emptyset$.

Moreover, G is closed convex and invariant under T; that is, $T(G) \subset G$. Therefore, $G \in \Phi$ is a lower bound for ψ. Then by Zorn's lemma, Φ has a minimal element, say C_0.

If C_0 is a singleton, say $\{x_0\}$, proof is complete since $T(C_0) \subset C_0$ would then imply $Tx_0 = x_0$.

Let $\overline{co}(TC_0)$ denote the closed convex hull of $T(C_0)$. Since $T(C_0) \subset C_0$, we have $\delta(\overline{co}(C_0)) = d > 0$. Since C has normal structure, there exists a point $x_0 \in C_0$ which is not diametral, i.e., there exists $B(x_0, d_1)$ such that $0 < d_1 < d$ and $C_0 \subset B(x_0, d_1)$.

Let $F = \{x \in C_0 : C_0 \subset B(x, d_1)\} = C_0 \cap \{\cap_{y \in C_0} B(y, d_1)\}$.

Then $F \subset C_0$. $F \neq C_0$ since $d_1 < d$. Now F is closed and convex. That F is invariant under T follows as given here. If $x \in F \subset C_0$ and $y \in C_0$, by the nonexpansiveness of T one gets, $\|Tx - Ty\| \leq \|x - y\| \leq d_1$, so that $TC_0 \subset B(Tx, d_1)$. But $B(Tx, d_1)$ is a closed convex set containing TC_0, so $C_0 = \overline{co}(TC_0) \subset B(Tx, d_1)$ and by the definition of F, one have $Tx \in F$.

Thus, F is a closed convex invariant proper subset of C_0 contradicting the minimality of C_0 in Φ. Therefore, C_0 contains exactly one point.

Remark 2.2.1. Theorem 2.2.2 remains true if X is any Banach space and C is a convex weakly compact subset having normal structure.

It is worth noting that all the conditions of Theorem 2.2.2 are necessary. In order to have a better insight into the theorem we demonstrate the same using examples which are essentially borrowed from Singh et al [149].

(a) In order to demonstrate the closedness of C, let $X = \Re$ be a Hilbert space and C the interior of the unit ball; that is, $C = \{x : \|x\| < 1\}$. Define $T : C \to C$ by $Tx = \frac{1}{2}(x + \alpha)$ where $\|\alpha\| = 1$, α is real. Obviously, T is nonexpansive, but T has no fixed point.

(b) A translation in a Banach space is an isometry and obviously has no fixed point which establishes the necessity of C to be bounded in Theorem 2.2.2.

(c) For exhibiting the essentiality of convexity of C, let $X = \Re$ be a Hilbert space and C the set containing just the two points a and b. Define $T : C \to C$ such that $Ta = b$, $Tb = a$. Clearly T is an isometry, but has no fixed points.

(d) For demonstrating the necessity of reflexivity of X let $X = C[0, 1]$ be a Banach space with sup norm. It is known that $C[0, 1]$ is not a reflexive Banach space. Let $C = \{f(t) \in C[0, 1] : 0 = f(0) \leq f(t) \leq f(1) = 1\}$. Define $T : C \to X$ by $Tf(t) = t.f(t)$, $t \in [0, 1]$. Then $T(C) \subset C$ and T has no fixed point.

It is worth mentioning that one of the most revealing question still remains unanswered: Is reflexivity essential for f.p.p.? However, there is compelling evidence that the answer might be affirmative. First it is known that some closed bounded convex sets in the classical nonreflexive spaces c_0 and l_1, fail to have f.p.p.. Also, it has been shown that if X is any Banach space with an unconditional basis, then X is reflexive if and only if X contains a subspace isomorphic to c_0 and l_1.

(e) For establishing the necessity of normal structure of X let c_0 be the Banach space of null sequences and let C be the unit ball in c_0. Define the mapping $T : C \to C$ by $T(x_1, x_2, \cdots) = (1, x_1, x_2, \cdots)$. Then T is a nonexpansive mapping from C to itself but has no fixed points since $(x_1, x_2, x_3, \cdots) = (1, x_1, x_2, \cdots)$ would imply that $x_1 = x_2 = x_3... = 1$ and, hence, $(x_1, x_2, \cdots) \notin c_0$. In this case, the Banach space $X = c_0$ does not have normal structure.

An immediate consequence of above theorem is the following well-known result, which was proved independently by Browder [19], Göhde [66] and Kirk [87].

Theorem 2.2.3. Every nonexpansive self mapping T of a closed bounded and convex subset of a uniformly convex Banach space has a fixed point.

Tasković [161, 2002] extended Theorem 2.2.2 to diametral contraction.

Definition 2.2.1. Let C be a nonempty subset of a Banach space X. A mapping $T : C \to X$ is called *diametral contraction* on C if

$$\|Tx - Ty\| \leq \sup\{\|x - z\| : z \in C\} \ \ \forall\, x, y \in C.$$

Definition 2.2.2. Let $\mathcal{F}$ be a collection of all closed bounded and convex subsets of a normed space X. A mapping $T : C \to C$ (for $C \in \mathcal{F}$) is said to be *diametral contractive* if

$$\|Tx - Ty\| \leq \sup\{\|x - z\| : z \in K\}$$

for every $K \in \mathcal{F}$ with $K \subset C$ and for all $x, y \in K$. A normed space X is said to have *diametral fixed point property* if every diametral contractive mapping $T : C \to C$ has a fixed point.

Theorem 2.2.4. Let C be a nonempty closed bounded and convex subset of a reflexive Banach space X and suppose that C has normal structure. If T is a diametral contractive mapping of C into itself, then T has a fixed point in C.

Since the reflexivity of the space and normal structure of C are consequences of the uniform convexity of X, following corollary can be derived directly.

Corollary 2.2.1. Let C be a nonempty closed bounded and convex subset of a uniformly convex Banach space X and T be a self mapping of C. If there are real numbers $a, b, c \geq 0$ such that

$$\|Tx - Ty\| \leq a\|x - y\| + b\|x - Tx\| + c\|x - Ty\|$$

and $a + b + c = 1$ for all $x, y \in C$, then T has a fixed point in C.

As an immediate consequence of Theorem 2.2.4 one can obtain the following statements:

Corollary 2.2.2. Let C be a nonempty closed bounded and convex subset of a reflexive Banach space X and suppose that C has normal structure. If T maps C into itself and if there are real numbers $a, b, c \geq 0$ such that

$$\|Tx - Ty\| - a\|x - y\| \leq \max\{b\|x - Tx\| + c\|x - Ty\|, b\|y - Ty\| + c\|y - Tx\|\}$$

and $a + b + c = 1 \ \ \forall\, x, y \in C$, then T has a fixed point in C.

Corollary 2.2.3. Let C be a nonempty closed bounded and convex subset of a reflexive Banach space X and suppose that C has normal structure. If T is a mapping of C into itself such that

$$\sup\{\|Tx - Ty\| : y \in C\} \leq \sup\{\|x - y\| : y \in C\} \ \ \forall\, x \in C,$$

then T has at least one fixed point in C.

Question 2.2.1. Does Theorem 2.2.4 remains true for reflexive Banach spaces without normal structure?

Question 2.2.2. If every normed space X have diametral fixed point property does X has normal structure?

Belluce and Kirk [6] extended Theorem 2.2.2 for finite family of commuting nonexpansive mappings in following way.

Theorem 2.2.5. If C is a nonempty bounded weakly compact and convex subset of a Banach space X and if C has normal structure then every finite family $\mathcal{F}$ of commuting nonexpansive mappings of C into itself has a common fixed point.

Here it may be pointed out that Theorem 2.2.5 holds for infinite families if the norm of X is strictly convex. For if the norm is strictly convex then the fixed point set for each $T \in \mathcal{F}$ is nonempty closed bounded and convex. Hence these fixed point sets are weakly compact and have the finite intersection property. Thus there will be a point common to all of them.

On the other hand, same authors [7] later showed that by strengthening the condition of normal structure appropriately one can establish the existence of a common fixed point for arbitrary families without assuming strict convexity of the norm.

Definition 2.2.3. For subsets C and K of X with C bounded, let (for $x \in X$)

$$\begin{aligned} r_x(C) &= \sup\{\|x-y\| : y \in C\}, \\ r(C,K) &= \inf\{r_x(C) : x \in K\}, \\ \mathcal{C}(C,K) &= \{x \in K : r_x(C) = r(C,K)\}. \end{aligned}$$

The set $\mathcal{C}(C,K)$ is frequently referred to as the *Chebyshev center of C in X.*

Definition 2.2.4. Let C be a closed bounded convex subset of X. Then C is said to have *complete normal structure* if every closed convex subset K of C which contains more than one point satisfies the condition that "for every decreasing net $\{K_\alpha : \alpha \in \wedge\}$ of subsets of K which have the property that $r(K_\alpha, K) = r(K,K), \alpha \in A$ it is the case that the closure of $\bigcap_{\alpha\in\wedge}\mathcal{C}(K_\alpha, K)$ is a nonempty proper subset of K".

Theorem 2.2.6. Suppose C is a weakly compact convex subset of a Banach space X, such that C is equipped with complete normal structure. Let $\mathcal{F}$ be a commuting family of nonexpansive mappings of C into itself. Then there is a point $x \in C$ such that $T(x) = x$ for each $T \in \mathcal{F}$.

Kirk [89] obtained conditions sufficient to guarantee the existence of fixed points for a mapping T such that T^n is nonexpansive.

Theorem 2.2.7. Let X be a reflexive Banach space equipped with strictly convex norm and suppose C is a nonempty closed bounded and convex subset of X which possesses normal structure. Suppose the mapping $T : C \to C$ has the property that for some integer $n > 1$, T^n is nonexpansive and suppose further that there is a constant k satisfying

$$n^{-2}[(n-1)(n-2)k^2 + 2(n-1)k] < 1 \tag{2.2.1}$$

such that $\|T^j x - T^j y\| \leq k\|x - y\|$ for all $x, y \in C, 1 \leq j \leq n-1$. Then T has a fixed point in C.

For the case $n = 2$, inequality (2.2.1) becomes $k < 2$. Goebel [59] obtained a sharper result in this case using the concept of "modulus of convexity". Specifically, let B be the closed unit ball in X and let

$$\delta(\epsilon) = \inf\{1 - \|x - y\|/2 : x, y \in B \text{ and } \|x - y\| \geq \epsilon\}$$

with X and C as in Theorem 2.2.7, then a mapping $T : C \to C$ has a fixed point if $\|T^2 x - T^2 y\| \leq \|x - y\|$ where

$$(k/2)(1 - \delta(2/k)) < 1.$$

The above inequality holds for $k < 2$ in an arbitrary Banach space. This yields Theorem 2.2.7 in the case $n = 2$ in arbitrary spaces.

Several other positive results were proved under various conditions of geometric type on the norm of X ensuring the existence of a fixed point for a nonexpansive mapping T. The following notions and results are crucial in describing the forthcoming results,

(a) (Browder [18]) X admits a sequentially continuous duality function $F_\phi : X, \sigma(X, X^*) \to X^*, \sigma(X^*, X)$ (i.e., a function F_ϕ such that $\langle x, F_\phi(x)\rangle = \|x\|\|F_\phi(x)\|$ and $\|F_\phi(x)\| = \phi(\|x\|)$ for all $x \in X$, where $\phi : \Re^+ \to \Re^+$ is continuous strictly increasing with $\phi(0) = 0$ and $\phi(+\infty) = +\infty$.

(b) (Opial [120]) If a sequence $\{x_n\}$ converges weakly in X to x_0, then

$$\liminf \|x_n - x\| > \liminf \|x_n - x_0\|, \quad \forall\, x \neq x_0.$$

(c) (Brodskii and Milman [14]) Every weakly compact convex subset C of X has normal structure.

When T is single-valued, the existence of a fixed point for T in C was proved by Browder [18] if X satisfies (a) and if T can be extended outside C in a nonexpansive way and by Kirk [87] if X satisfies (c).

In 1972, Gossez and Dozo [68] not only proved the relationship between conditions (a), (b) and (c) but also defined following two more conditions:

(*d*) same as (*a*) except that F_ϕ is only required to be sequentially continuous at zero and

(*e*) same as (*b*) except that $>$ is replaced by $\geq$ to establish the further connection between the above defined geometric properties.

Theorem 2.2.8. (*a*) implies (*b*) and (*b*) implies (*c*). No converse implication holds, even when X and X^* are assumed to be uniformly convex.

Theorem 2.2.9. (*d*) implies (*e*). The converse implication holds when the norm of X is uniformly Gâteaux differentiable.

The last part of Theorem 2.2.8 can be explained as follows:

When $1 < p < \infty$ $p \neq 2$, $L^p(0, 2\pi)$ satisfies (*c*) since it is uniformly convex but Opial [120] showed that even (*e*) does not hold. When $1 < p \neq q < \infty$ the Hilbert product of l^p and l^q satisfies (*b*) but Bruck [30] showed that (*a*) does not hold. A finite dimensional space whose norm is not differentiable provides another example of a space satisfying (*d*) and (*b*) but not (*a*).

The following situation holds in general:

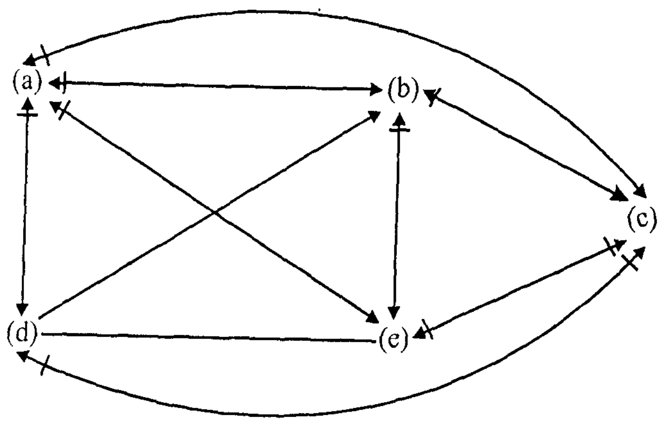

Fig. 2.2.1

Following examples exhibits a space satisfying (*d*) but not (*c*).

Example 2.2.1. Consider the space l^p endowed with the norm

$$\|x\| = \max\left\{\frac{1}{2}\|x\|_{l^2}; \|x\|_{l^\infty}\right\}.$$

$$x = (x_1, x_2, \ldots) \to \begin{cases} \frac{x}{4} & \text{if } \frac{1}{2}\|x\|_{l^2} \geq \|x\|_{l^\infty} \\ (0, \ldots.., 0, x_{m_0}, 0, \ldots.) & \text{if } \frac{1}{2}\|x\|_{l^2} < \|x\|_{l^\infty}, \end{cases}$$

where $m_0 = \inf\{m : x_m = \|x\|_{l^\infty}\}$ and x_{m_0} stands at the m_0^{th} place, defines a duality function which is sequentially weakly continuous at zero.

Example 2.2.2. Consider the sequence space c_0 with the following norm

$$\|x\| = \sup\left\{\sum_{i=1}^{\infty} 2^{-2i} x_{\alpha_i}^2\right\}^{1/2}$$

where the supremum is taken over all permutations α of N, defines on c_0 an equivalent norm [35] which is known to be locally uniformly convex. c_0 endowed with this norm satisfies (d) but not (c).

Although (e) does not generally imply (b), following theorem due to Gossez and Lami [68] demonstrates the condition under which it does.

Theorem 2.2.10. (e) implies (b) when X is uniformly convex.

In 1973, Kannan [83] substituted the condition of normal structure on the subset C of X with the *Property C* on mapping T in Theorem 2.2.2 described as follows:

Definition 2.2.5. For every nonempty closed bounded and convex T-invariant subset K of C which contains more than one element there exists $x \in K$ such that

$$\sup_{r} \|x - T^r x\| < \sup_{z,y \in K} \|z - Ty\|.$$

It should be noted that if K has normal structure then T has *Property C* on K where T is a nonexpansive mapping of K into itself. Converse is not true as it can be easily seen from the following example.

Example 2.2.3. Let B be the space isomorphic to the Hilbert space X with norm defined by

$$\|x\| = \sup\left\{\frac{1}{2}\|x\|_X, |x_n|\right\} \qquad x \in X.$$

Then $C = \{x : \|x\|_X \leq 1 \text{ and } x_i \geq 0 \text{ for all } i\}$ does not have normal structure. It is easy to define nonexpansive mapping T on C for which Property C is true.

Using 'Property C', Kannan [83] gave the following modified version of Theorem 2.2.2.

Theorem 2.2.11. Let T be a continuous self mapping of a closed bounded subset C of a reflexive Banach space X into itself such that T enjoys *property C* over C. If T satisfies either

$$\|Tx - Ty\| \leq \frac{1}{2}\{\|x - Tx\| + \|y - Ty\|\}, \quad x, y \in C,$$

or

$$\|Tx - Ty\| \leq \|x - y\|, \quad x, y \in K,$$

then T has a fixed point in C.

Remark 2.2.2. Theorem 2.2.10 can be partially derived from Theorem 2.2.2.

Gillespie and Williams [56] introduced the concept of uniform normal structure in Banach spaces and proved that

Theorem 2.2.12. Any nonexpansive self mapping of a closed bounded and convex subset C of a Banach space X with uniformly normal structure has a fixed point.

Also Gillespie and Williams [57] proved a fixed point theorem for certain Kannan-type mapping under the assumption that X has uniform normal structure, which is as follows:

Theorem 2.2.13. If C is a closed bounded and convex subset of a Banach space X and $T : C \to C$ such that

$$\|Tx - Ty\| \leq \frac{1}{2}\{\|x - Tx\| + \|y - Ty\|\},$$

then T has a unique fixed point provided X possesses uniform normal structure.

Since it is known that a Banach space X having uniform normal structure is always reflexive hence the main results of Gillespie and Williams [56, 57] are contained in Kirk [87], Göhde [66] and Kannan [83] respectively. In an attempt to improve Theorem 2.2.13, Bose and Sahani [12] proved the following fixed point theorem.

Theorem 2.2.14. Let C be a closed bounded and convex subset of a Banach space X equipped with a uniform normal structure. If $T : C \to C$ is such that

$$\|Tx - Ty\| \leq a\|x - y\| + b\{\|x - Tx\| + \|y - Ty\|\} + c\{\|x - Ty\| + \|y - Tx\|\}$$

for all $x, y \in C$ and for some $a, b, c \geq 0$ such that $3a + 2b + 4c \leq 1$, then T has a fixed point.

The initial attempts to extend Theorem 2.2.2 to the nonlinear cases were not very successful. In 1979, Penot's [124] formulation of this theorem may be considered as a gateway to some interesting new results. Specially, the extension of normal structure ideas to some discrete sets.

Penot [124] presented an abstract version of Theorem 2.2.2 for nonexpansive mappings. Penot observed that a fairly straightforward modification of the proof of Theorem 2.2.2 and yields the following:

Theorem 2.2.15. If (X, d) is a bounded metric space equipped with a convexity structure which is compact and normal, then every nonexpansive mapping

$T : X \to X$ has a fixed point.

Penot's [124] result is based essentially upon the original line of argument which uses Zorn's lemma. Subsequently, Kirk [93] proved that a different approach yields the abstract result under even weaker assumptions. In general Kirk [93] showed that countable compactness suffices for this result.

Theorem 2.2.16[93]. Let S be a class of countably compact subsets of a bounded metric space (X, d) stable under arbitrary intersection and normal. Suppose further that S contains the closed balls of X. Then every nonexpansive mapping T of X into itself has a fixed point.

The above theorem differs from Penot's result(cf. [124]) wherein countable compactness is assumed rather than compactness. Later Kulesza and Lim [98] showed that a normal convexity structure is compact if and only if it is countably compact.

Browder [22] applied general theory of accretive operators to strengthen the fixed point theory of nonexpansive mappings in uniformly convex spaces.

Definition 2.2.6. Let X be a Banach space. If T is a mapping with domain $D(T)$ in X and with values in X, then T is said to be *accretive* if for all $u, v \in D(T)$ and some $j \in J(u - v)$,

$$\langle Tu - Tv, j \rangle \geq 0,$$

where j denotes the normalized duality mapping.

In case f is a nonexpansive mapping of $D(f)$ into X and if we set $T = I - f$, $D(T) = D(f)$, then T is an accretive mapping of $D(T)$ into X.

Theorem 2.2.17. Let X be a Banach space equipped with uniform structure, B a closed ball in X, C an open subset of X containing B. Suppose that T is a nonexpansive mapping of C into X which maps the boundary of B into B. Then T has a fixed point in B.

Similar fixed point theorems for nonexpansive mappings, in spaces which do not have normal structure can be found in Browder [22, 26], Browder and Petryshyn [29], Petryshyn and Williamson [127, 128], Kirk [89, 90, 92] and references quoted therein.

Browder and Petryshyn [29] proved the following.

Theorem 2.2.18. Every nonexpansive self mapping of a nonempty closed bounded and convex subset of a Hilbert space X has a fixed point.

The unbounded closed convex set in Banach spaces fall into two distinct categories. The one category is consisted of those which are linearly unbounded in the sense that they have an unbounded intersection with at least one line, whereas, the other category is consisted of those sets which are linearly bounded.

Any convex set C in a reflexive Banach space which is unbounded yet linearly bounded possesses "almost fixed point property" i.e., $\inf\{\|x - Tx\| : x \in C\} = 0$ for each nonexpansive mapping $T : C \to C$. Special cases of this result are available in Goebel and Kuczumow [65] and Ray [130] and it was established in full generality by Reich [134]. In this regard the following question remains open.

Question 2.2.3. Does there exist an unbounded closed and convex subset of a Banach space which has the fixed point property for nonexpansive mapping?

If such a set exists it must of course be linearly bounded. Sine [148] showed that no such set can exist in Hilbert spaces.

In case X is a Hilbert space and C is closed convex unbounded and linearly unbounded, then it is not difficult to construct a fixed point free mapping. For this we have

Example 2.2.4. Let T be a self mapping defined on a linearly bounded set $C = \{x \in l^1 : \|x\| \leq 1 \text{ for all } i\}$ such that $T : (x_1, x_2, \ldots) \to (1, x_1, x_2, \ldots)$. Clearly, T is an isometry which is fixed point free.

Ray [131] proved the following theorem for fixed point property of nonexpansive mappings in Hilbert space.

Theorem 2.2.19. Let C be a closed and convex set in a real Hilbert space X. Then C has the f.p.p. for nonexpansive mappings if and only if C is bounded.

Since the set $F(T)$, set of fixed points of T, may be empty unless X has some geometrical and topological properties. In 1981, Goebel and Koter [64] proved a fixed point theorem by putting some additional conditions on the mapping T itself.

Definition 2.2.7. A nonexpansive mapping $T : X \to X$ is called *n-rotative* if there exists an integer $n \geq 2$ and a real number $a < n$ such that for any $x \in X$

$$\|x - T^n x\| \leq a\|x - Tx\|.$$

It should be noted that if T is nonexpansive, then for each positive integer m, $\|x - T^m x\| \leq m\|x - Tx\|$. One can observe that if T is n-rotative, it is also m-rotative for $m > n$.

The simplest examples of rotative nonexpansive mappings are all contractions, rotations of Euclidean space $\Re^n$ or any periodic nonexpansive mapping in any Banach space.

Theorem 2.2.20. The set of fixed points of a rotative nonexpansive self mapping defined on a Banach space is nonempty.

§ 2.3. Results via iterations

It is well known that unlike contraction mappings sequence of iterates need not converge for nonexpansive self mapping T defined on a nonempty closed bounded and convex subset C of a Banach space X. Karsnoseleskii [85] proved that, for some $x_0 \in C$, the sequence $\{T^n x_0\}$ does not converge necessarily to a fixed point of T whereas the sequence $\{T_\alpha^n x_0\}$, where

$$T_\alpha = (1-\alpha)I + \alpha T, \quad 0 < \alpha \leq 1 \tag{2.3.1}$$

may converge to a fixed point of T. Karsnoselskii [85] proved that the sequence $\{T_{\frac{1}{2}}^n x_0\}$ associated with mapping T defined on a compact convex subset C of a uniformly convex Banach space X converges to a fixed point T. Schaefer [142] extended this result for a general number α.

The scheme (2.3.1) extended by the so-called "Mann iterative process" associated with a self mapping T of a normed space X is described as

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n$$

for $x_0 \in X$ and $n > 0$ where $\{\alpha_n\}$ satisfies the conditions

(a) $\alpha_0 = 1$,

(b) $0 \leq \alpha_n < 1$ and

(c) $\sum \alpha_n$ diverges.

Mann [111] proved that if a continuous self mapping T has a unique fixed point in a closed unit interval, then the Mann iterates converge to that fixed point. By now this method of Mann [111] is one of the most powerful methods to approximate fixed points in Banach spaces. Mann's iteration method have been investigated by several authors such as: Krasnoselskii [85], Edelstein [47], Opial [120], Outlaw [122], Dotson [42] and others. They showed that these iterative methods may be used to prove a fixed point theorems for a nonexpansive mapping T in Banach space equipped with specific geometric properties (e.g. uniform convexity, strict convexity etc.). In what follows, we mention some selected fixed point theorems of this kind.

Outlaw [122] discussed the iteration procedure introduced by Mann [111] on compact convex sets in a strictly convex Banach space and gave the following result.

Theorem 2.3.1[122]. Let C be a compact convex subset of a strictly convex Banach space X and let T be a nonexpansive mapping on C. Let $x_1 \in C$ and define a sequence $\{x_n\}$ in C by

$$x_{n+1} = \frac{1}{2}Tx_n + \frac{1}{2}x_n.$$

Then $\{x_n\}$ converges strongly to a fixed point of T.

Ishikawa [77, 1976] extended this theorem to a Banach space without strict convexity. Ishikawa [77] considered the following iterative procedure which is a special case of the generalized iteration method introduced by Mann [111].

Definition 2.3.1. If C is a subset of a Banach space X, T is a mapping from C into X and $x_1 \in C$, then $M(x_1, \alpha_n, T)$ is the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n,$$

where $\{\alpha_n\}$ is a real sequence. If a point x_1 and sequence $\{\alpha_n\}$ satisfy the following three conditions:

(a) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$,

(b) $0 \leq \alpha_n \leq b < 1$ for all positive integer n,

(c) $x_n \in C$ for all positive integer n,

then x_1 and $\{\alpha_n\}$ will be said to satisfy *condition* (A).

Ishikawa [77] employed his iterative method to prove results on nonexpansive mappings without any assumption of convexity on subsets of underlying Banach spaces. Following is a fixed point theorem for a nonexpansive mapping which illustrates how the iterative process $M(x_1, \alpha_n, T)$ may be utilized to realize the fixed point.

Theorem 2.3.2. Let C be a closed subset of a Banach space X and let T be a nonexpansive mapping from C into a compact subset of X. If there exist x_1 and $\{\alpha_n\}$ that satisfy condition (A), then T has a fixed point in C and $M(x_1, \alpha_n, T)$ converges to a fixed point of T.

Following corollaries are immediate from Theorem 2.3.2.

Corollary 2.3.1. Let C be a closed subset of a Banach space X and let T be a nonexpansive mapping from C into a compact subset of X. If there exists $\alpha \in (0, 1)$ such that $(1 - \alpha)x + \alpha T x \in C$ for all $x \in C$, then T has a fixed point in C and for any $x_1 \in C$, $M(x_1, \alpha, T)$ converges to a fixed point of T.

Corollary 2.3.2. Let C be a closed convex subset of a Banach space X and let T be a nonexpansive mapping from C into a compact subset of X. Then T has a fixed point in C and $M(x_1, 2^{-1}, T)$ converges to a fixed point of T for any $x_1 \in C$.

Here, it may be pointed out that the first part of Corollary 2.3.2 is a special case of a fixed point theorem of Schauder [143]. Corollary 2.3.2 was proved for uniformly convex spaces by Krasnoselskii [85] whereas for strictly convex spaces by Edelstein [47].

Next, Ishikawa [77] proved the iterative process for a nonexpansive mapping without the assumption on the compactness of T.

Definition 2.3.2[146]. Let C be a subset of a Banach space X. A mapping $T : C \to X$ with a nonempty fixed points set $F(T)$ in C will be said to satisfy *Condition B* if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(t) > 0$ for $t \in (0, \infty)$, such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in C$, where $d(x, F(T)) = \inf\{\|x - z\| : z \in F(T)\}$.

Theorem 2.3.3. Let C be a closed subset of a Banach space X and let $T : C \to X$ be nonexpansive mapping with a nonempty fixed points set $F(T)$ in C. If T satisfies *Condition B* and there exist x_1 and $\{\alpha_n\}$ that satisfy *condition A*, then $M(x_1, \alpha_n, T)$ converges to a member of $F(T)$.

Corollary 2.3.3. Let C be a closed convex subset of a Banach space X and let $T : C \to C$ be a nonexpansive mapping with a nonempty fixed points set $F(T)$. If T satisfies *Condition B*, then, for any $x_1 \in C$ and any $\{\alpha_n\}$ satisfying (a) and (b) of Definition 2.3.1, $M(x_1, \alpha_n, T)$ converges to a member of $F(T)$.

Reich [133] discussed the iteration procedure introduced by Mann [111] in a uniformly convex Banach spaces whose norm is Fréchet differentiable.

Theorem 2.3.4. Let C be a closed convex subset of a uniformly convex Banach space X with a Fréchet differentiable norm. If $T : C \to C$ is a nonexpansive mapping with a fixed point, then $\{T^n x\}$ is weakly almost convergent to a fixed point of T.

Theorem 2.3.5. Let X be a uniformly convex Banach space X with a Fréchet differentiable norm, let C be a closed convex subset of X and $T : C \to C$ is a nonexpansive mapping with a fixed point. Let $x_1 \in C$ and define a sequence $\{x_n\}$ in C by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that

$$\sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = \infty.$$

Then $\{x_n\}$ converges weakly to a fixed point of T.

In 1974, Ishikawa [76] introduced a new iteration process as follows

$$x_{n+1} = \alpha_n T(\beta_n T x_n + (1 - \beta_n)x_n) + (1 - \alpha_n)x_n, \qquad (2.3.2)$$

$n = 1, 2, 3\ldots$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ satisfying the following conditions

(a) $0 \leq \alpha_n \leq \beta_n \leq 1$ for all positive integer n,

(b) $\lim_{n\to\infty} \beta_n = 0$,

(c) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$.

Note that Ishikawa [76] iteration reduces to Mann iteration if one set $\beta_n = 0$ for all n.

Tan and Xu [159] proved the following interesting result which generalizes the result of Reich [133], i.e., Theorem 2.3.5 using the iterative scheme introduced by Ishikawa [76].

Theorem 2.3.6. Let C be a closed convex subset of a uniformly convex Banach space X which satisfies Opial's condition or whose norm is Fréchet differentiable and let $T : C \to C$ be a nonexpansive mapping with a fixed point. Then for any $x_1 \in C$, the iterates $\{x_n\}$ defined by (2.3.2) converge weakly to a fixed point of the involved mapping T provided $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen so that

$$\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty, \quad \sum_{n=1}^{\infty} \beta_n(1 - \alpha_n) < \infty \text{ and } \lim_{n\to\infty} \sup \beta_n < 1.$$

The following theorem due to Deng [39] extends Theorem 2.3.5 of Reich [133] to uniformly convex Banach spaces.

Theorem 2.3.7. Let X be a uniformly convex Banach space with a Fréchet differentiable norm. If T is a nonexpansive self mapping of a closed convex subset of a Banach space X with a fixed point, then the sequence $\{x_n\}$ described by (2.3.2) converges to a fixed point of T.

Zeng [170] proved a result which is a complement of Theorem 2.3.6 and generalizes Theorem 2.3.5 to certain extent.

Theorem 2.3.8. Let X be a uniformly convex Banach space that satisfies the Opial's condition or whose norm is Fréchet differentiable. Let C be a bounded closed convex subset of X and $T : C \to C$ be a nonexpansive mapping. Then for any $x_1 \in C$ the iterates $\{x_n\}$ described by (2.3.2) converges weakly to a fixed point of T provided that $\lim_{n\to\infty} \sup \beta_n < 1$ and for any subsequence $\{n_k\}$ of $\{n\}$, $\sum_{k=0}^{\infty} \alpha_{n_k}(1 - \alpha_{n_k})$ diverges.

Remark 2.3.1. Note that Theorem 2.3.8 reduces to Theorem 2.3.5 if we set $\beta_n = 0$ for all n.

Zeng [170] also gave a strong convergence theorems utilizing Ishikawa iteration scheme (2.3.2).

Theorem 2.3.9. Suppose that X is a uniformly convex Banach space and T, C, and $\{x_n\}$ are as in Theorem 2.3.8. Moreover if $T(C)$ is contained in a compact subset of X, then the Ishikawa iterates $\{x_n\}$ described by (2.3.2) converges strongly to a fixed point of T.

Theorem 2.3.10. Let X be a uniformly convex Banach space and let T and C be as in Theorem 2.3.8. If T satisfies Condition B, then for any $x_1 \in C$, the Ishikawa iteration process $\{x_n\}$ described by (2.3.2) provided that $\lim\limits_{n\to\infty} \sup \beta_n < 1$ and $\sum\limits_{n=0}^{\infty} \alpha_n(1-\alpha_n)$ diverges, converges strongly to a fixed point of T.

Takahashi and Kim [158] also proved the following theorem.

Theorem 2.3.11. Let X be a uniformly convex Banach space which satisfies Opial's condition or whose norm is Fréchet differentiable, let C be a nonempty closed convex subset of X and $T : C \to C$ be a nonexpansive mapping with a fixed point. The sequence $\{x_n\}$ of Ishikawa iterates described by (2.3.2) converges weakly to a fixed point of T.

Where the involved constants $\{\alpha_n\}$ and $\{\beta_n\}$ in $\{x_n\}$ satisfy $\alpha_n \in [a, b]$ and $\beta_n \in [0, b]$ or $\alpha_n \in [a, 1]$ and $\beta_n \in [a, b]$ for some a, b with $0 < a \leq b < 1$.

Then motivated by Theorem 2.3.6 and Theorem 2.3.11, Suzuki and Takahashi [156] obtained the following theorem.

Theorem 2.3.12. Let C be a closed convex subset of a uniformly convex Banach space X which satisfies Opial's condition or whose norm is Fréchet differentiable. Let T be a nonexpansive mapping on C with a fixed point. The sequence of Ishikawa iterates $\{x_n\}$ described by (2.3.2) converges weakly to a fixed point of T where the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in the context of $\{x_n\}$ satisfy

$$\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty \quad \text{and} \quad \lim_{n\to\infty} \sup \beta_n < 1$$

or

$$\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \sup \beta_n < 1.$$

Comparision of Theorem 2.3.12 with Theorem 2.3.6 indicates that the assumption $\sum\limits_{n=1}^{\infty} \beta_n(1-\alpha_n) < \infty$ in Theorem 2.3.6 is superfluous.

The following strong convergence theorem due to Shioji et al. [152] is connected with Tan and Xu [159] and Takahashi and Kim [158].

Theorem 2.3.13. Let C be a nonempty closed convex subset of a strictly convex Banach space X and $T : C \to C$ be a nonexpansive mapping such that $T(C)$ is contained in a compact subset of C. Then the sequence $\{x_n\}$ defined by (2.3.2) converges strongly to a fixed point of T where $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen so that

$$\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty \quad \text{and} \quad \limsup_{n\to\infty} \beta_n < 1,$$

or

$$\sum_{n=1}^{\infty} \beta_n(1-\beta_n) = \infty \quad \text{and} \quad \liminf_{n\to\infty} \alpha_n > 0.$$

In 1996, Deng [39] generalized Theorem 2.3.2 using Ishikawa iteration process and established weak and strong convergence results for nonexpansive mappings in Banach spaces.

Theorem 2.3.14. Let C be a nonempty subset of a normed linear space X and $T : C \to X$ be a nonexpansive mapping. Let $\{x_n\}$ be a sequence of Ishikawa iterates defined by (2.3.2) in C, $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of real numbers such that

(*a*) $0 \leq \alpha_n \leq a < 1$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$,

(*b*) $0 \leq \beta_n \leq 1$ and $\sum\limits_{n=1}^{\infty} \beta_n < \infty$,

(*c*) $\{x_n\}$ is bounded.

Then $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$.

Theorem 2.3.15. Let X be a Banach space which satisfies Opial's condition. Let C be a weakly compact and $T : C \to X$ be a nonexpansive mapping. Let $\{x_n\}$ be a sequence of Ishikawa iterates defined by (2.3.2) in C whereas $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of real numbers such that (a) and (b) of above theorem are satisfied. Then $\{x_n\}$ converges to a fixed point of T.

The two following theorems generalize Theorem 2.3.2 and Theorem 2.3.3 due to Ishikawa [77] respectively.

Theorem 2.3.16. Let C be a closed subset of a Banach space X and let T be a nonexpansive mapping from C into a compact subset of X. Let $\{x_n\}$ be a sequence of Ishikawa iterates defined by (2.3.2) in C whereas $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of real numbers such that (a) and (b) of Theorem 2.3.14 are satisfied, then $\{x_n\}$ converges to a fixed point of T.

Theorem 2.3.17. Let X, C and $\{x_n\}$ be same as in Theorem 2.3.16. Let $T : C \to X$ be a nonexpansive mapping whose set of fixed points $F(T)$ is nonempty in C. If

T satisfies Condition B, then $\{x_n\}$ converges to a fixed point of T.

Kang et al. [82] in their result replaced condition (b) of Theorem 2.3.13 by

$$\sum_{n=1}^{\infty} \alpha_n \beta_n < \infty \text{ and } \beta_n \to 0 \quad \text{as } n \to \infty \text{ or } \beta_n = b \; \forall \, n \geq 1 \quad \text{and } \beta \in [0,1)$$

and proved that Theorem 2.3.17 still remains true.

Theorem 2.3.18[82]. Let C be a nonempty subset of a normed linear space X and $T : C \to X$ be a nonexpansive mapping. Let $\{x_n\}$ be a sequence in C of Ishikawa Iterates defined by (2.3.2), $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers such that

(a) $0 \leq \alpha_n \leq a < 1$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$,

(b) (i) $0 \leq \beta_n \leq 1$ and $\lim\limits_{n\to\infty} \beta_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n \beta_n < \infty$ or

(ii) $\beta_n = b$ for all $n \geq 1$ and $b \in [0,1)$.

If $\{x_n\}$ is bounded, then

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

In order to prove Theorem 2.3.18 following lemmas and definition are required.

Lemma 2.3.1[82]. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ be three sequences of nonnegative real numbers satisfying the inequality

$$\alpha_{n+1} \leq (1 + \delta_n)\alpha_n + \beta_n.$$

If $\sum\limits_{n=1}^{\infty} \delta_n < \infty$ and $\sum\limits_{n=1}^{\infty} \beta_n < \infty$, then $\lim\limits_{n\to\infty} \alpha_n$ exists. In particular, if the sequence $\{\alpha_n\}$ has a subsequence converging to zero, then $\lim\limits_{n\to\infty} \alpha_n = 0$.

Lemma 2.3.2[82]. Let X be a normed linear space and C be a nonempty subset of X. Let $T : C \to X$ be a nonexpansive mapping. Let $\{x_n\}$ be a sequence in C satisfying (2.3.2). Then we have

$$\|x_{n+1} - Tx_{n+1}\| \leq (1 + 2\alpha_n\beta_n)\|x_n - Tx_n\|, \quad \forall \, n \geq 1.$$

Definition 2.3.3. A metric space (X, d) is said to be of *hyperbolic type* if X contains a family L of metric segments such that

(a) each two points $x, y \in X$ are endpoints of exactly one member segment of L,

(b) if p, x, $y \in X$ and $m \in \text{seg}[x, y]$ satisfies $d(x, m) = \alpha d(x, y)$ for $\alpha \in [0, 1]$, then

$$d(p, m) \leq (1 - \alpha)d(p, x) + \alpha d(p, y).$$

Lemma 2.3.3[82]. Let (X, d) be of hyperbolic type and $\{\alpha_n\}$ be a sequence in $[0, 1)$. If $\{x_n\}$ and $\{y_n\}$ be sequences in X such that, for all $n \geq 1$,

(a) $x_{n+1} \in \mathrm{seg}[x_n, y_n]$ with $d(x_n, x_{n+1}) = \alpha_n d(x_n, y_n)$,

(b) $d(y_{n+1}, y_n) \leq d(x_{n+1}, x_n)$,

(c) $d(y_i, x_i) \leq M < \infty \ \forall\, i, n \geq 1$,

(d) $\alpha_i \leq \alpha < 1$,

(e) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$,

then $\lim\limits_{n\to\infty} d(y_n, x_n) = 0$.

Proof of Theorem.

First assume that $\sum\limits_{n=1}^{\infty} \alpha_n\beta_n < \infty$ and $\beta_n \to 0$ as $n \to \infty$. By Lemmas 2.3.1 and 2.3.2, we see that $\|x_n - Tx_n\|$ exists, say d. Setting $a_n = Tx_n - x_n$, then one have $\|a_n\| \to d$ as $n \to \infty$. Without loss of generality, one may assume that $\alpha_n > 0$ for all $n \geq 1$. Otherwise, consider a subsequence $\{\alpha_j\}$ of $\{\alpha_n\}$. Setting

$$b_n = \alpha_n^{-1}(Tx_{n+1} - Tx_n) + Tx_n - Ty_n,$$

we have $a_{n+1} = (1 - \alpha_n)a_n + \alpha_n b_n$. Following the proof of Deng [39], one can get the following conclusions:

(a) $\limsup\limits_{n\to\infty} \|b_n\| \leq d$,

(b) $\|\sum\limits_{i=1}^{n} \alpha_i b_i\| \leq \|x_{n+1} - x_1\| + \sum\limits_{i=1}^{n} \alpha_i\beta_i\|Tx_i - x_i\|$ is bounded.

Thus the conclusion of Theorem 2.3.18 follows from Deng [39]

Next, assume that $\beta_n = \beta$ for all $n \geq 1$ and $\beta \in [0, 1)$. In order to complete the proof of this theorem, one need to verify the conditions (a) to (e) of Lemma 2.3.3.

The condition (a) is obvious and the conditions (d) and (e) are natural.
The proofs of conditions (b) and (c) runs as follows:

Setting $z_n = (1 - \beta)x_n + \beta Tx_n$ and $y_n = Tz_n$, then we have

$$\begin{aligned} d(y_{n+1}, y_n) &= \|Tz_{n+1} - Tz_n\| \\ &\leq \|z_{n+1} - z_n\| \\ &= \|(1 - \beta)x_{n+1} + \beta Tx_{n+1} - (1 - \beta)x_n - \beta Tx_n\| \\ &\leq (1 - \beta)\|x_{n+1} - x_n\| + \beta\|x_{n+1} - x_n\| \end{aligned}$$

$$= \|x_{n+1} - x_n\| = d(x_{n+1}, x_n)$$

which verifies the condition (b).

The following

$$\begin{aligned} d(y_{i+n}, x_i) &= \|Tz_{i+n} - x_i\| \\ &\le \|Tz_{i+n} - Tx_{i+n}\| + \|Tx_{i+n} - x_i\| \\ &\le \|z_{i+n} - x_{i+n}\| + \|Tx_{i+n} - x_i\| \\ &\le \beta\|x_{i+n} - Tx_{i+n}\| + \|Tx_{i+n} - x_i\| \le M < \infty, \end{aligned}$$

which verifies the condition (c).

By Lemma 2.3.3, one assert that $\lim_{n\to\infty} \|Tz_n - x_n\| = 0$. Observe that

$$\begin{aligned} \|Tx_n - x_n\| &\le \|Tx_n - Tz_n\| + \|Tz_n - x_n\| \\ &\le \|x_n - z_n\| + \|Tz_n - x_n\| \\ &= \beta\|x_n - Tx_n\| + \|Tz_n - x_n\|, \end{aligned}$$

which implies that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. This completes the proof.

As an immediate consequence of Theorem 2.3.18, we have the following:

Corollary 2.3.4. Let T be a nonexpansive self mapping of a nonempty compact convex subset C of a real normed space X. If $\{x_n\}$ be a sequence in C as defined in Theorem 2.3.18, then $\{x_n\}$ converges strongly to a fixed point of T.

Corollary 2.3.5. Let T be a nonexpansive self mapping of a nonempty closed convex subset C of a real Banach space X with the nonempty fixed point set $F(T)$. Let the sequence $\{x_n\}$ in C be as in Theorem 2.3.18. If T satisfies Condition B, then $\{x_n\}$ converges strongly to a fixed point of T.

Theorem 2.3.19. Let C be weakly compact subset of a Banach space X which satisfies Opial's condition and $T : C \to X$ be a nonexpansive mapping. Let $\{x_n\}$ be a sequence of Ishikawa iterates defined by (2.3.2) in C, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers such that

(*a*) $0 \le \alpha_n \le \alpha < 1$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(*b*) (i) $0 \le \beta_n \le 1$ and $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n\beta_n < \infty$ or

(ii) $\beta_n = \beta$ for all $n \ge 1$ and $\beta \in [0, 1)$.

Then $\{x_n\}$ converges weakly to a fixed point of T.

§ 2.4. Structure of the set of fixed points

The fixed points set $F(T)$ of a nonexpansive mapping T in general is neither convex nor connected nor weakly closed (see, Gruber [69] and Sine [147]). Goebel and Kirk [63] proved that for X is strictly convex, $F(T)$ is (closed and) convex.

Theorem 2.4.1. The set $F(T)$ of fixed points of a nonexpansive self mapping defined on a closed and convex subset of a strictly convex Banach space is closed and bounded.

Proof.
The set of fixed points $F(T)$ is closed because T is continuous. Suppose $x = Tx$ and $y = Ty$. Let $\lambda \in (0,1)$ and set $z = (1-\lambda)x + \lambda y$. Then

$$\begin{aligned} \|x - Tz\| + \|Tz - y\| &= \|Tx - Tz\| + \|Tz - Ty\| \\ &\leq \|x - z\| + \|z - y\| \\ &= \|x - y\| \\ &\leq \|x - Tz\| + \|Tz - y\|. \end{aligned}$$

It follows that x, Tz and y are collinear while $\|x - z\| = \|x - Tz\|$ and $\|y - z\| = \|y - Tz\|$. Since X is strictly convex, $z = Tz$.

The above fact does not hold in general Banach spaces.

Example 2.4.1. Assign $\Re^2$ with the norm $\|(x,y)\|_\infty = \max\{|x|, |y|\}$ and define $T : \Re^2 \to \Re^2$ by $T(x,y) = (x, |x|)$.

Clearly, T is nonexpansive relative to $\|.\|_\infty$ and $F(T)$ is the graph $\{(x,y) : y = |x|\}$, which is not convex. In fact, if $f : \Re \to \Re$ is any function satisfying

$$|f(t) - f(s)| \leq |t - s|,$$

then the graph of f is the fixed point set of the nonexpansive mapping on $(\Re^2, \|.\|_\infty)$ defined by $T(x,y) = (x, f(x))$.

Following example shows that the set $F(T)$ may even be disconnected.

Example 2.4.2. Consider the unit ball in the space c_0. Define the nonexpansive mapping $T : c_0 \to c_0$ by $Tx = T(x_1, x_2, ...) = (x_1, 1 - |x_1|, x_2, x_3, ...)$.

Since $\lim\limits_{n\to\infty} x_n = 0$, if $x = Tx$ then $x_2 = x_3 = ... = 0$. Thus $|x_1| = 1$ and $F(T)$ consists of the two points e^1 and $-e^1$.

Definition 2.4.1. A metric space (X, d) is said to be metrically convex if for any two points $x, y \in X$ with $x \neq y$ there exists $z \in X$, $x \neq z \neq y$ such that

$$d(x, z) + d(z, y) = d(x, y).$$

In nonstrictly convex setting $F(T)$ need not be convex. However, for a wide class of Banach spaces, the fixed point sets of nonexpansive mappings are metrically convex (hence, in particular always connected).

Definition 2.4.2. A Banach space X is said to have the f.p.p. for spheres if each nonempty closed and convex subset of the unit sphere has the f.p.p. for nonexpansive mappings.

Theorem 2.4.2. Suppose C is a nonempty closed convex subset of a Banach space having f.p.p. for spheres and suppose that $T : C \to C$ is nonexpansive. Then $F(T)$ is metrically convex.

Khamsi [84, 1989] proved that a space X is strictly convex if and only if for any nonexpansive mapping T on any convex set, $F(T)$ is convex. For another characterization of strict convexity in terms of nonexpansive mappings, see Müller and Reinermann [115, 1979], wherein it was proved that convexity of a set is implied by the validity of a property slightly stronger than f.p.p. of the involved nonexpansive mapping.

In Lin and Sternfeld [107, 1985], a characterization of compactness for convex sets in terms of f.p.p. for Lipschitz mappings was given. A similar result concerning starshapedness and finite dimensional spaces was indicated in Müller and Reinermann [115].

The existing literature concerning fixed point theory for nonexpansive mappings is so wide, that it is almost impossible to attempt a complete survey. Even we find hard to choose the most relevant results and condense them in few pages. The present literature also contains results in spaces with richer structure for Banach lattices which are available in Nelson, Singh and Whitfield [118, 1987] and Aksoy and Khamsi [1, 1990].

For results concerning fixed points for isometries see Lau [100, 1980]. Also results for other classes of sets were considered: balls (see, Nadler [117, 1981] for Euclidean spaces), etc. Some topological results concerning fixed points and polyhedra (in finite dimensional spaces) were indicated by Thomeier [162]. Here it may be pointed that we have not made any attempt to peep into purely topological fixed point theorems but our aim here is to have a bird eye view of the available metric fixed point theory for nonexpansive mappings.

* * * * *

CHAPTER 3

MULTI-VALUED NONEXPANSIVE MAPPINGS

§ 3.1. Introduction

The results concerning the existence of fixed points for contractions and nonexpansive mappings can be at least partially extended to the case of multi-valued mappings. In this chapter, the emphasis is principally on nonexpansive mappings. Recall (Def. 1.5.1) that by a multi-valued mapping we mean a mapping of the form $T : X \to 2^X$; thus T assigns to each $x \in X$ a nonempty subset of X. For multi-valued mappings the notion of fixed point is modified, $x \in X$ is said to be a fixed point of T if and only if x is contained in its image set, i.e., $x \in Tx$. More generally we state the f.p.p. for multi-valued mapping as follows:

Definition 3.1.1. A set X is said to have the *f.p.p.* for a family of multi-valued mappings if each member of the family has a fixed point.

Nadler [116] introduced the notion of multi-valued Lipschitz mappings can be defined as follows:

Definition 3.1.2. Let (X, d_X) and (Y, d_Y) be two metric spaces and $T : X \to CB(Y)$ be a multi-valued mapping. The mapping T is said to be a *multi-valued Lipschitz mapping* if for all $x, y \in X$ there exists a constant k such that

$$d_Y(Tx, Ty) \leq k d_X(x, y).$$

As before, T is called *multi-valued contraction* if $k < 1$ and T is said to be *nonexpansive* if $k = 1$.

The study of fixed point theorems for multi-valued mappings was initiated by Kakutani [81] in 1941, wherein he proved that every upper semicontinuous multi-valued mappings defined on a compact convex set C in $\Re^n$ and taking values in $KK(C)$ has a fixed point. Since then various well known results on fixed point were extended to multi-valued mappings which generally associate each point of the metric space (X, d) to a closed subset of X. Kakutani's work [81] was extended to infinite dimensional Banach spaces by Bohnenblust and Karlin [10] in 1950 and to locally convex topological vector space by Ky Fan [51] in 1952 and till today the fixed point theory in functional analysis for multi-valued mappings has been extensively developed. The geometric fixed point theory for multi-valued mappings was initiated and studied by Nadler [116] and subsequently pursued by Markin [112], Assad and Kirk [3], Browder [25, 27], Lami-Dozo [99] and many others.

Fixed point theorems for multi-valued mappings are useful in control theory and have been effectively used in tackling problems in economics and game theory.

§ 3.2. Fixed point theorems

In [112], Markin proved the following fixed point theorem for multi-valued mappings, which is an extension of a theorem of Browder [17, 1965].

Theorem 3.2.1. Let B be a closed unit ball in a Hilbert space X and $T : X \to C(X)$ be a nonexpansive mapping. Then T has a fixed point in B provided $T(x) \subset B$ for every $x \in B$.

Subsequently, Lami-Dozo [99] considered the following interesting case where the Banach space is assumed to satisfy the Opial's condition. This result generalizes results those of Markin [112] and Browder [17].

Theorem 3.2.2. Let C be a nonempty weakly compact convex subset of a Banach space X which satisfies Opial's condition, then every nonexpansive multi-valued mapping $T : C \to \text{cpt}(C)$ defined on C whose values are nonempty compact subsets of C possesses a fixed point.

Proof.

Let $x_0 \in C$ be a fixed element. Assume that $\{k_n\}$ be a sequence in $(0,1)$ which converges to 1. Define

$$T_n x = k_n Tx + (1 - k_n)x_0. \tag{3.2.1}$$

Then $T_n : C \to \text{cpt}(X)$ and each T_n is a contraction. By Nadler Contraction Principle (Theorem 1.5.8), there exists an $x_n \in C$ such that $x_n \in T_n x_n$. Since C is weakly compact, there exists a subsequence of $\{x_n\}$, again denoted by $\{x_n\}$, converging weakly to $x \in C$. From (3.2.1), we deduce

$$x_n = k_n z_n + (1 - k_n)x_0, \text{ where } z_n \in Tx_n.$$

So, $\|x_n - z_n\| = (1 - k_n)\|x_0 - z_n\|$.

Hence, $y_n = x_n - z_n \in (I - T)x_n$ and $y_n \to 0$ as $n \to \infty$.

Which implies that $(x_n, y_n) \in G(I - T)$, where $G(I - T)$ denotes the graph of $I - T$, with $x_n \to x$ and $y_n \to 0$. Since $I - T$ is demiclosed $0 \in (I - T)x$, i.e., $x \in Tx$.

Remark 3.2.1. If Banach space X is restricted to be a Hilbert space one can obtain Theorem 3.2.1 and if Banach space is considered with a weakly continuous duality mapping one can obtain a result due to Browder [17] because these spaces satisfy Opial's condition (Def. 1.2.18).

Assad and Kirk [3] obtained an improved version of Lami-Dozo [99] generalization of Theorem 3.2.1. Before stating the result due to Assad and Kirk [3] we state the following result which will be used in proof of the next theorem.

Theorem 3.2.3[3]. Let C be a closed convex subset of a Banach space X and K a closed subset of C. If T be a contraction mapping from K into the family of all nonempty closed bounded subsets of C such that $T(x) \subset K$ whenever $x \in \delta_C K$, then T has a fixed point in K.

Note. $\delta_C K$ denotes the boundary of K relative to C. In particular, if K is closed

$$\delta_C K = \{z \in K : B(z,r) \cap C/K \neq \emptyset \text{ for each } r > 0\},$$

where $\quad B(z,r) = \{x \in X : \|z - x\| < r\}.$

Theorem 3.2.4[3]. Let C be a closed convex subset of a Banach space X which satisfies Opial's condition and K a nonempty weakly compact convex subset of C. Let $T : K \to \mathrm{cpt}(C)$ be a nonexpansive multi-valued mapping on K into the nonempty compact subsets of C and suppose $Tx \subset K$ whenever $x \in \delta_C K$. Then T has a fixed point in K.

Proof.

Without loss of generality, it can be assumed that $0 \in K$. Let $\{k_n\}$ be a sequence of real numbers in (0,1) such that $k_n \to 1$ as $n \to \infty$. For each n, $k_n T$ is a multi-valued contraction mapping form K to the nonempty compact subsets of C. Furthermore, if $x \in \delta_C K$ then $k_n Tx \subset K$ because $0 \in K$ and K is convex. By Theorem 3.2.3, it follows that for each n, $k_n T$ has a fixed point in K; say $x_n \in k_n T x_n \cap K, n = 1, 2, \ldots$. Hence $x_n/k_n \in Tx_n$ and thus $x_n(1 - 1/k_n) \in x_n - Tx_n = (I - T)x_n$. Since K is weakly compact and $\{x_n\} \subset K$, it follows that $\{x_n\}$ has a weakly convergent subsequence, again represented by $\{x_n\}$ which converges, say to x_0. Furthermore, since $\{x_n\}$ is bounded,

$$w_n = x_n(1 - 1/k_n) \to 0 \text{ (strongly)}.$$

Following the argument of Lami-Dozo [99], $0 \in (I - T)x_0$.

Since $w_n \in (I - T)x_n$, one may write $w_n = x_n - u_n$ where $u_n \in Tx_n$.
Thus

$$H(Tx_n, Tx_0) \leq \|x_n - x_0\|,$$

and $u_n \in Tx_n$ implies there exists $\overline{u_n} \in Tx_0$ such that

$$\|u_n - \overline{u_n}\| \leq H(Tx_n, Tx_0).$$

Thus

$$\|u_n - \overline{u_n}\| \leq \|x_n - x_0\|.$$

It follows that

$$\liminf_{n\to\infty} \|x_n - x_0\| \geq \liminf_{n\to\infty} \|u_n - \overline{u_n}\|$$

$$= \liminf_{n\to\infty} \|x_n - w_n - \overline{u_n}\|.$$

Now, since $\{\overline{u_n}\}$ is contained in the compact set Tx_0, one can suppose subsequences again have been chosen so that $\{\overline{u_n}\}$ converges strongly, say to $u_0 \in Tx_0$. Therefore,

$$\begin{aligned}
\liminf_{n\to\infty} \|x_n - w_n - \overline{u_n}\| &= \liminf_{n\to\infty} \|x_n - w_n - \overline{u_n} + u_0 - u_0\| \\
&\geq \liminf_{n\to\infty} [\|x_n - u_0\| - \|w_n\| - \|\overline{u_n} - u_0\|] \\
&\geq \liminf_{n\to\infty} \|x_n - u_0\| + \liminf_{n\to\infty} (-\|w_n\|) \\
&\quad + \liminf_{n\to\infty} (-\|\overline{u_n} - u_0\|) \\
&= \liminf_{n\to\infty} \|x_n - u_0\|.
\end{aligned}$$

Thus

$$\liminf_{n\to\infty} \|x_n - x_0\| \geq \liminf_{n\to\infty} \|x_n - u_0\|.$$

Since $x_n \to x_0$ weakly, therefore by Opial's condition, $x_0 = u_0$. Hence the theorem is proved.

Itoh and Takahashi [78] gave the following common fixed point theorem for single-valued nonexpansive mapping and multi-valued nonexpansive mapping in Banach spaces that satisfy Opial's condition. Their result also extends Theorem 3.2.2.

Theorem 3.2.5. Let f be an asymptotically regular nonexpansive mapping of C into C, where C is a nonempty weakly compact convex subset of a Banach space X which satisfies Opial's condition and T be a multi-valued nonexpansive mapping of C into 2^C such that for any $x \in C$, $T(x)$ is nonempty compact. Then there exits an element $z \in C$ such that $f(z) = z \in T(z)$ provided f and T commute.

Itoh and Takahashi [78] also gave some fixed point theorems for multi-valued mappings defined on a starshaped subsets of a Banach space which are as follows:

Theorem 3.2.6. Let C be a weakly compact starshaped subset of a Banach space X which satisfies Opial's condition. Let T be a nonexpansive mapping of C into 2^X, where for any $x \in C$, $T(x)$ is nonempty compact and for each $x \in \delta C$, $T(x) \subset C$. Then T has a fixed point in C.

For single-valued mappings the following corollary can be obtained.

Corollary 3.2.1. Let C be a weakly compact starshaped subset of a Banach space X which satisfies Opial's condition. Let T be a nonexpansive mapping of C into X such that $T(\delta C) \subset C$, then there exists a fixed point of T in C.

If C is compact, then the following theorem holds in any Banach space.

Theorem 3.2.7[78]. Let C be a compact starshaped subset of a Banach space X and T be a nonexpansive mapping of C into 2^X, where for each $x \in C$, $T(x)$ is nonempty compact and for any $x \in \delta C$, $T(x) \subset C$. Then T has a fixed point in C.

As Corollary 3.2.1, one can again obtain a corollary to Theorem 3.2.7 for single-valued mappings.

Corollary 3.2.2. Let C be a compact starshaped subset of a Banach space X and T be a nonexpansive mapping of C into X such that $T(\delta C) \subset C$, then there exists a fixed point of T in C.

Yanagi [169] gave extension of Theorem 3.2.2 and Theorem 3.2.4 by using the concept of inward sets.

Definition 3.2.1. Let C be a nonempty subset of a Banach space X. For $x \in C$, *inward set of x relative to C*, denoted by $I_C(x)$, is defined as follows:

$$I_C(x) = \{x + \alpha(y - x) : y \in C, \alpha \geq 1\}.$$

A mapping $T : C \to CB(X)$ is said to be *inward* if $Tx \subseteq I_C(x)$ $\forall\ x \in C$ and *weakly inward* if $Tx \subseteq \overline{I_C(x)}$ $\ \forall\ x \in C$.

Theorem 3.2.8[169]. Let C be a nonempty weakly compact convex subset of a Banach space X and let $T : C \to \mathrm{cpt}(X)$ be nonexpansive and weakly inward mapping. If $I - T$ is demiclosed or semiconvex on C, then T has a fixed point.

Corollary 3.2.3. Let C be a nonempty weakly compact convex subset of a Banach space X which satisfies Opial's condition (or weak Opial's condition). If $T : C \to \mathrm{cpt}(X)$ is nonexpansive (or generalized contraction) and weakly inward, then T has a fixed point.

Proof.

Since X satisfies Opial's condition and T is nonexpansive, it implies that $I - T$ is demiclosed. Hence the result follows from Theorem 3.2.8.

If C is compact in the Theorem 3.2.8, one can obtain the following.

Corollary 3.2.4. Let C be a nonempty compact convex subset of a Banach space X and let $T : C \to \mathrm{cpt}(X)$ is nonexpansive such that $T(x) \subset \overline{I_C(x)}$ for each $x \in C$. Then T has a fixed point.

Furthermore, Yanagi [169] obtained fixed point theorem for nonexpansive mappings with nonconvex domains.

Theorem 3.2.9. Let C be a nonempty weakly compact starshaped subset of a uniformly convex Banach space X and let $T : C \to \text{cpt}(X)$ be nonexpansive. If for each $x \in \delta C$, $T(x) \subset C$ and $\alpha x + (1 - \alpha)T(x) \subset C$ for some $\alpha \in (0, 1)$ or $T(x) \subset \text{int}(C)$, then T has a fixed point.

On the lines of Lami-Dozo [99], Massa et al. [114] proved some results in non-convex setting in which T satisfies the following condition.

$$\text{for all} \quad x \in C, (x, y] \cap C \neq \emptyset \; \forall \, y \in Tx, \tag{3.2.2}$$

where $(x, y] = \{(1 - \alpha)x + \alpha y, 0 < \alpha \leq 1\}$.

Theorem 3.2.10. Let C a nonempty closed starshaped subset of a Banach space X. Let $T : C \to 2^X$ be a nonexpansive mapping such that the condition (3.2.2) holds. Further, assume that $T(C)$ is bounded and $(I - T)C$ is closed, then T has a fixed point.

Theorem 3.2.11. Let C a weakly closed starshaped subset of a Banach space X satisfying Opial's condition. Let $T : C \to 2^X$ be a nonexpansive mapping satisfying the condition (3.2.2) and let $T(C) \subseteq K$ for some weakly compact subset K of X. Then T has a fixed point.

In a parallel development to Assad and Kirk [3], several other important results were given, out of which following are worthy of attention.

Ko [96] proved the following important theorem for multi-valued nonexpansive mapping defined on a semiconvex subset of a Banach space.

Theorem 3.2.12[96]. Let C be a nonempty weakly compact convex subset of a Banach space X. If $T : C \to C(X)$ is nonexpansive. Then T has a fixed point in C provided $I - T$ is semiconvex on C.

Proof of the Theorem 3.2.12 highly relies upon the following theorem and lemma.

Theorem 3.2.13[96]. Let C be a nonempty closed weakly compact and convex subset of a Banach space X. If $T : C \to 2^C$ is upper semicontinuous and

$$\inf\{d(x, Tx) : x \in C\} = 0$$

and $I - T$ is a semiconvex mapping on C, then T has a fixed point in C.

Lemma 3.2.1. Let C be a nonempty closed bounded and convex subset of a Banach space X. If $T : C \to CB(X)$ is nonexpansive, then $\inf\{d(x, Tx) : x \in C\} = 0$.

Proof.

The mapping T is nonexpansive, so it is upper semicontinuous. Theorem 3.2.12 follows Theorem 3.2.13 provided that the condition "$\inf\{d(x, Tx) : x \in C\} = 0$" is satisfied. This condition follows from the Lemma 3.2.1.

The fixed point theory of multi-valued nonexpansive mappings is however much more complicated and difficult than the corresponding theory of single-valued nonexpansive mappings. One breakthrough was achieved by Lim [104] in 1974, wherein he proved a fixed point theorem by using Edelstein's method of asymptotic center [48]. Before stating the theorem we give the notion of asymptotic radius and asymptotic center.

Definition 3.2.2. Let C be a weakly compact convex subset of a Banach space X and $\{x_n\}$ a bounded sequence in X. Let T be a mapping define on X by

$$T(x) = \lim_{n\to\infty} \sup \|x_n - x\|, \ \forall\, x \in X.$$

Let

$$r = r_C(\{x_n\}) = \inf\{T(x) : x \in C\}$$

and

$$A = A_C(\{x_n\}) = \{x \in C : T(x) = r\}.$$

The numbers r and A, respectively, called the *asymptotic radius* and *asymptotic center* of $\{x_n\}$ relative to C. As C is weakly compact convex, $A_C(\{x_n\})$ is nonempty weakly compact and convex.

Theorem 3.2.14[104]. Let C be a nonempty closed bounded and convex subset of a uniformly convex Banach space X. Let $T : C \to \mathrm{cpt}(C)$ be a nonexpansive mapping, then T has a fixed point.

Remark 3.2.2. Theorem 3.2.14 remains true if X is required only to be reflexive and uniformly convex in every direction [see 36, 54]. Since in such a space the asymptotic center of a bounded sequence in a closed convex set is unique [103].

The original proof of Theorem 3.2.14 was later simplified independently by Lim himself [105] and Goebel [60]. Downing and Kirk [45] in 1977, obtained an inward version of Theorem 3.2.14 as follows:

Theorem 3.2.15. Let C a nonempty closed bounded and convex subset of a uniformly convex Banach space X and T a nonexpansive multi-valued mapping defined on C and taking values in the family of nonempty compact subsets of X. If $Tx \subset I_C(x)$ for all $x \in C$, then T has a fixed in C.

Reich [132] proved the following result from which the improved version of Lim's result can be derived as a corollary.

Theorem 3.2.16. Let C be a closed convex subset of a Banach space X and let T be a contraction mapping which assigns to each x in C a nonempty compact subset of X. If T is weakly inward, then T has a fixed point.

This result yields the following improvement of Lim [104].

Corollary 3.2.5. Let C be a weakly compact convex subset of a Banach space X which is uniformly convex in every direction. Let T be a mapping which assign to each $x \in C$ a nonempty compact subset of X. If T is nonexpansive inward mapping, then T has a fixed point.

Kirk and Massa [94] in 1990, also gave an extension of Theorem 3.2.14 proving the existence of a fixed point in a Banach space for which the asymptotic center of a bounded sequence in a closed bounded convex subset is nonempty and compact.

Theorem 3.2.17. Let C be a nonempty closed bounded and convex subset of a Banach space X and $T : C \to C(C)$ a nonexpansive mapping. Suppose that the asymptotic center in C of each bounded sequence of X is nonempty and compact. Then T has a fixed point.

Theorem 3.2.17 applies to all k-uniformly rotund Banach spaces [154]. However it does not apply to a nearly uniformly convex Banach space [73] as in such a space the asymptotic center of a bounded sequence is not necessarily compact. Also note that Theorem 3.2.17 requires that T take convex values.

Question 3.2.1. Does the conclusion of Theorem 3.2.17 remains valid if T only takes compact values?

The answer is affirmative if the space X is uniformly convex [104] or X satisfies Opial's property. But in general, the question is still unanswered.

Definition 3.2.3. Let C be a weakly compact convex subset of a Banach space X and $\{x_n\}$ a bounded sequence in X. Then $\{x_n\}$ is called *regular* w.r.t. C if $r_C(\{x_n\}) = r_C(\{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$; while $\{x_n\}$ is called *asymptotically uniform* if $A_C(\{x_n\}) = A_C(\{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$.

The method of asymptotic centers plays an important role in the fixed point theory for single- and multi-valued nonexpansive mappings. One can see this fact by following fundamental lemma.

Lemma 3.2.2[60, 105]. Let C be a weakly compact convex subset of a Banach space X and $\{x_n\}$ a bounded sequence in X. Then we have

(*i*) there always exists a subsequence of $\{x_n\}$ which is regular w.r.t. C,

(*ii*) if C is separable, then $\{x_n\}$ contains a subsequence which is asymptotically uniform w.r.t. C.

Remark 3.2.3[168]. If X is uniformly convex in every direction (especially uniformly convex), then $A_C(\{x_n\})$ consists of exactly one point so every regular sequence in such a space is always asymptotically uniform w.r.t. C.

Let now C be a weakly compact convex subset of a Banach space X and $T : C \to \mathrm{cpt}(C)$ a nonexpansive mapping. For each integer $n \geq 1$, the contraction $T_n : C \to \mathrm{cpt}(C)$ defined by

$$T_n(x) = \frac{1}{n}x_0 + \left(1 - \frac{1}{n}\right)Tx, \ \forall\, x \in C$$

where $x_0 \in C$ is a fixed point of T, has a fixed point $x_n \in C$. Let r and A be the asymptotic radius and center of $\{x_n\}$ w.r.t. C, respectively. It is easily seen that

$$d(x_n, Tx_n) \leq \frac{1}{n}\delta(C) \to 0.$$

Since T is compact valued, one can take $y_n \in Tx_n$ such that

$$\|y_n - x_n\| = d(x_n, Tx_n), \ \forall\, n \geq 1.$$

Since T is a self mapping, one may assume that C is separable (otherwise, one can construct a closed convex subset of C, that is, invariant under T(see [97])). Then by Lemma 3.2.2, one may assume that $\{x_n\}$ is asymptotically uniform. Take any $z \in A$, as Tz is compact, one can find $z_n \in Tz$ satisfying

$$\|y_n - z_n\| = d(y_n, Tz) \leq H(Tx_n, Tz).$$

It follows from the nonexpansiveness of T that

$$\|y_n - z_n\| \leq \|x_n - z\|.$$

Since Tz is compact, one may also assume that $\{z_n\}$ converges(strongly) to a point $\overline{z} \in Tz$. It then follows that

$$\limsup \|x_n - \overline{z}\| = \limsup \|y_n - z_n\| \leq \limsup \|x_n - z\|.$$

This shows that $\overline{z} \in A$. Hence one can define a multi-valued self map $\overline{T} : A \to A$ by setting for each $z \in A$,

$$\overline{T}z = A \cap Tz.$$

This mapping $\overline{T}$ is in general neither nonexpansive nor lower semicontinuous. However, it is upper semicontinuous, which is observed by Kirk and Massa [94]. With this observation they were able to prove Theorem 3.2.17 by using the Bohnenblust-Karlin [10] fixed point theorem that is of topological rather than metric nature. In addition, if one assume that X is uniformly convex (uniformly convex in every direction is enough), then asymptotic center A consists of exactly one point z. Then the above argument shows that one must have $\overline{z} = z$ and therefore z is a fixed point of T. This is the idea of the simplified proof of Lim's theorem given independently by Goebel [60] and Lim [105].

The two following results are now basic in the fixed point theory of multi-valued mappings. Which are proved by Deimling [37], and Downing and Kirk [45] and Reich [132] respectively.

Theorem 3.2.18[37]. Let C be a nonempty closed subset of a Banach space X and $T : C \to 2^X$ a contraction. Assume that T is weakly inward on C and that each $x \in C$ has a nearest point in Tx. Then T has a fixed point.

Theorem 3.2.19[45, 132]. Let C be a nonempty closed bounded and convex subset of a uniformly convex Banach space X and $T : C \to \text{cpt}(X)$ a nonexpansive mapping. Then T has a fixed point provided T is inward on C.

The following consequence of Theorem 3.2.18 above of Deimling [37] will be used in our next theorem.

Theorem 3.2.20. Let C be nonempty closed bounded and convex subset of a Banach space and $T : C \to C(X)$ a contraction. Assume $Tx \cap \overline{I_C(x)} \neq \emptyset$, for all $x \in C$. Then T has a fixed point.

Xu [168] in 2001 gave an extension of Kirk and Massa's [94] result to nonself mappings utilizing the idea of universal net.

Definition 3.2.4. A net $\{x_\alpha\}$ in a set S is called *universal net* if for each subset U of S, either $\{x_\alpha\}$ is eventually in U or $\{x_\alpha\}$ eventually in S/U.

The following facts are pertinent (see Def. 2.2, [168]).

(a) Every net in a set has a universal subnet.

(b) If $T : S_1 \to S_2$ is a map and if $\{x_\alpha\}$ is a universal net in S_1, then $\{T(x_\alpha)\}$ is a universal net in S_2.

(c) If S is compact and if $\{x_\alpha\}$ is a universal net in S, then $\lim\limits_{\alpha} x_\alpha$ exists.

Now for a nonself mapping $T : C \to \text{cpt}(X)$, one can define the contraction by the same formula (Remark 3.2.3). Suppose for each integer $n \geq 1$, T_n has a fixed point $x_n \in C$. Let $\{x_{n_\alpha}\}$ be a universal subnet of $\{x_n\}$. Xu [168] work out the asymptotic center as above just by replacing the sequence $\{x_n\}$ by the net $\{x_{n_\alpha}\}$.

This technique was employed by Xu [168] to give the following extension of Theorem 3.2.17.

Theorem 3.2.21. Let C be a nonempty closed bounded and convex subset of a Banach space X and $T : C \to C(X)$ be a nonexpansive nonself mapping which satisfies the inwardness condition. Suppose that the asymptotic center in C of each bounded sequence of X is nonempty and compact. Then T has a fixed point.

Proof.

Fix $x_0 \in C$ and for each integer $n \geq 1$, define the contraction $T_n : C \to C(X)$ by

$$T_n(x) = \frac{1}{n}x_0 + \left(1 - \frac{1}{n}\right)Tx, \ \forall \, x \in C.$$

Then T_n satisfies the inwardness condition, i.e., $T_n(x) \subset I_C(x)$ for all $x \in C$. Thus by Theorem 3.2.19, T_n has a fixed point $x_n \in C$. By Lemma 3.2.2, one may assume that $\{x_n\}$ is regular. Let $y_n \in Tx_n$ be constructed as Remark 3.2.3, i.e., $\|x_n - y_n\| = d(x_n, Tx_n)$. Let $\{x_{n_\alpha}\}$ be a universal subnet of $\{x_n\}$ and define a function g by

$$g(x) = \limsup_{\alpha} \|x_{n_\alpha} - x\|, \ \forall \, x \in C.$$

Let

$$K = \{x \in C : g(x) = r\},$$

where $r = \inf\limits_{x \in C} g(x)$. Then by assumption, K is nonempty and compact. The key to proof is that the inwardness of T on C implies a weaker inwardness of T on K, i.e.,

$$Tx \cap I_K(x) \neq \emptyset, \quad x \in K. \tag{3.2.3}$$

Indeed, if $x \in K$, by compactness, for each $n \geq 1$, there exists some $z_n \in Tx$ such that

$$\|y_n - z_n\| = d(y_n, Tx) \leq H(Tx_n, Tx) \leq \|x_n - x\|.$$

Let $z = \lim\limits_{\alpha} z_{n_\alpha} \in Tx$. It follows that

$$\begin{aligned} g(z) &= \lim_{\alpha} \|x_{n_\alpha} - z\| \\ &= \lim_{\alpha} \|x_{n_\alpha} - z_{n_\alpha}\| \\ &\leq \lim_{\alpha} \|x_{n_\alpha} - x\|. \end{aligned}$$

Hence

$$g(z) \leq g(x) = r. \tag{3.2.4}$$

It remains to show $z \in I_K(x)$. As $Tx \subset I_C(x)$,

$$z = x + \lambda(v - x),$$

for some $\lambda \geq 0$ and $v \in C$.

If $\lambda \leq 1$, then by the convexity of C, $z \in C$ and hence by (3.2.4), $z \in K \subset I_K(x)$ and theorem is proved. So assume that $\lambda > 1$. Then one can write

$$v = \mu z + (1-\mu)x, \text{ where } \mu = \frac{1}{\lambda} \in (0,1).$$

By the convexity of g, one have by (3.2.4),

$$g(v) \leq \mu g(z) + (1-\mu)g(x) \leq r.$$

Since $v \in C$, it follows that $v \in K$ and thus $z = x + \lambda(v-x)$ belongs $I_K(x)$. Now, one have a nonexpansive mapping $T : K \to C(X)$ which satisfies the boundary condition (3.2.3). The following lemma shows that T has a fixed point in K.

Lemma 3.2.3. If C is a compact convex subset of a Banach space X and $T : C \to C(X)$ is a nonexpansive mapping satisfying the boundary condition

$$Tx \cap \overline{I_C(x)} \neq \emptyset \ \forall\, x \in C.$$

Then T has a fixed point.

Proof.

Fix an $x_0 \in C$ and for each integer $n \geq 1$, define a mapping $T_n : C \to C(X)$ by

$$T_n(x) = \frac{1}{n}x_0 + \left(1 - \frac{1}{n}\right)Tx, \ \forall\, x \in C.$$

Then T_n is a contraction satisfying the same boundary condition as T does, i.e., one have

$$T_n(x) \cap \overline{I_C(x)} \neq \emptyset, \quad \forall\, x \in C.$$

Hence by Theorem 3.2.20, T_n has a fixed point $x_n \in C$. Since C is compact, one may assume $x_n \to x \in C$. Also it is easily seen that

$$d(x_n, Tx_n) \leq \frac{1}{n}\delta(C) \to 0 \text{ as } n \to \infty.$$

Taking limit as $n \to \infty$ yields $d(x, Tx) = 0$ and hence $x \in Tx$.

Remark 3.2.4. If T satisfies a stronger condition: $Tx \cap I_C(x) \neq \emptyset$ for all $x \in C$, then Lemma 3.2.3 follows from a fixed point theorem of Caristi [32]. However, in nonexpansive case, the proof is constructive.

Remark 3.2.5. It is unclear whether the conclusion of Theorem 3.2.21 remains valid if the inwardness of T is weakened to the weak inwardness.

Theorem 3.2.22[168]. Assume C to be a closed bounded and convex subset of a uniformly convex Banach space X and let $T : C \to \text{cpt}(X)$ is a nonexpansive mapping satisfying the weak inwardness condition. Then T has a fixed point.

§ 3.3. Extension of multi-valued mappings

The notion of nonexpansiveness for multi-valued mappings has been extended in several ways. Husain and Tarafdar [75] introduced the concept of weakly nonexpansive mappings.

Definition 3.3.1. Let C be a nonempty subset of a normed space X, a multi-valued mapping $T : C \to 2^C$ is called *weakly nonexpansive* if given $x \in C$ and $u_x \in T(x)$ there is a $u_y \in T(y)$ for each $y \in C$ such that

$$\|u_x - u_y\| \leq \|x - y\|.$$

The notion of fixed point for a weakly nonexpansive mapping T remains the same, i.e., $x_0 \in C$ is called a fixed point of T if $x_0 \in T(x_0)$.

If $T_\alpha (\alpha \in \wedge)$ is a family of single-valued nonexpansive self mapping of C, then $T(x) = \cup_{\alpha \in \wedge} T_\alpha(x)$ $(x \in C)$ defines a weakly nonexpansive multivalued mapping.

Moreover, each single-valued mapping is nonexpansive if and only if it is weakly nonexpansive. To see that not every weakly nonexpansive mapping on a nonempty closed bounded and convex subset C of a Banach space X has a fixed point, we consider the following example.

Example 3.3.1[74]. Let $X = c_0$, the Banach space of all complex sequences converging to zero with the sup norm. Let C denote the closed unit ball of c_0. For each integer $n \geq 1$, define

$$T_n(x) = e_n + s_n(x) \quad \forall \, x \in C$$

where $e_n = \{\delta_{ni}\}_{i \geq 1}$ (the usual basis of c_0) and $s_n(x) = (x_1, x_2, ..., x_{n-1}, 0, x_{n+1}, ...)$ with 0 in the n^{th} position. It is easy to see that each $T_n : C \to C$ is a single-valued nonexpansive fixed point free mapping. The multi-valued mapping $T(x) = \cup_{n \geq 1} T_n(x)$ $(x \in C)$ is clearly weakly nonexpansive and has no fixed point. It may be noted that C is not weakly compact or equivalently X is not reflexive.

Husain and Tarafdar [75] proved the following result for weakly nonexpansive mappings.

Theorem 3.3.1. If C is a compact interval of the real line, then each weakly nonexpansive closed convex multi-valued mapping $T : C \to 2^C$ has a fixed point.

Husain and Latif [74] proved a fixed point theorem for weakly nonexpansive multi-valued mappings on nonempty convex weakly compact subsets of a Banach space under certain conditions since, as the above example shows that it is not true in general.

Theorem 3.3.2. Let C be a nonempty weakly compact convex subset of a Banach space X which satisfies Opial's condition. Let $T : C \to 2^C$ be a compact-valued weakly nonexpansive mapping. Assume the following holds:

(*) For a fixed $w \in C$ and $0 < k_n < 1$ with $k_n \to 0$, there is $u_x \in T(x)$ for all $x \in C$ such that each single-valued self mapping $T_n(x) = k_n u_x + (1 - k_n)w$ of C has a fixed point $x_n \in C$.

Then T has a fixed point.

Since each closed bounded and convex subset of a reflexive Banach space is weakly compact and each Hilbert space satisfies Opial's condition the following corollaries can be derived.

Corollary 3.3.1. Let C be a nonempty closed bounded and convex subset of a reflexive Banach space X (in particular, a uniformly convex Banach space) satisfying Opial's condition. Then each compact-valued weakly nonexpansive mapping $T : C \to 2^C$ satisfying $(*)$ has a fixed point.

Corollary 3.3.2. Let C be a nonempty closed bounded and convex subset of a Hilbert space X. Then each compact-valued weakly nonexpansive map $T : C \to 2^C$ satisfying $(*)$ has a fixed point.

Remark 3.3.1. Since for single-valued mappings, the concept of weakly nonexpansive maps coincides with that of nonexpansive mappings. Moreover, for single-valued mapping $T : C \to C$ where C is a nonempty closed bounded and convex subset of a Banach space X, each T_n in condition $(*)$ is a contraction map and so by Banach contraction principle, each T_n has a fixed point $x_n \in C$. Thus the assumption $(*)$ becomes redundant. Moreover, $\lim_{n\to\infty} \|T(x) - T_n(x)\| = 0$, i.e., each such T is the pointwise limit of contraction mappings.

"If C is a nonempty convex weakly compact subset of a Banach space X satisfying Opial's condition, then each single-valued nonexpansive mapping $T : C \to C$ has a fixed point". This is a consequence of a theorems due to Kirk [87] and Browder [19]. Theorem 3.2.2 includes this result as a special case.

Husain and Latif [74] introduced another class of nonexpansive mappings which is defined as follows:

Definition 3.3.2. Let C be a nonempty subset of a normed space X. A multi-valued map $T : C \to 2^C$ is said to be *-nonexapnsive if for all $x, y \in C$ and $u_x \in T(x)$

with

$$\|x - u_x\| = \inf\{\|x - z\| : z \in T(x)\},$$

there exists $u_y \in T(y)$ with

$$\|y - u_y\| = \inf\{\|y - w\| : w \in T(y)\}$$

such that

$$\|u_x - u_y\| \leq \|x - y\|.$$

Following is an example of a multi-valued *-nonexpansive mapping.

Example 3.3.2[74]. Consider $C = \{(a,b) \in \Re^2 : a^2 + b^2 \leq 1\}$. For each $x = (a,b)$, one define $T(x) = D_x$, where D_x is the unique diagonal perpendicular to the straight line joining the points (a,b) and $(0,0)$. Clearly $u_x = (0,0)$ for all $x \in C$ and so $0 = \|u_x - u_y\| \leq \|x - y\|$, showing that T is *-nonexpansive. It is easy to see that $(0,0)$ is the unique fixed point of T.

Husain and Latif [74] gave the following fixed point theorem for *-nonexpansive mappings.

Theorem 3.3.3. Let C be a nonempty closed bounded and convex subset of a uniformly convex Banach space X which satisfies Opial's condition. Then each closed convex valued *-nonexpansive mapping $T : C \to 2^C$ has a fixed point.

In the particular case when X is a Hilbert space we can dispense with the $(*)$ condition of Theorem 3.3.2 as well as Opial's condition and we have an improved fixed point theorem for *-nonexpansive mappings.

Corollary 3.3.3. Let C be a nonempty closed bounded and convex subset of a Hilbert space X. Then each convex closed valued *-nonexpansive mappings $T : C \to 2^C$ has a fixed point.

Corollary 3.3.3 extends the result of Husain and Tarafdar [75] and includes the result of Browder and Petryshyn [29] for single-valued mappings.

Xu [167] in 1991 gave some new fixed point theorems for *-nonexpansive multi-valued mappings wherein he showed that a weakly nonexpansive multi-valued mapping must be nonexpansive and thus the main theorem of Husain and Tarafdar [75] and of Hussain and Latif [74] on weakly nonexpansive multi-valued mappings are special cases of those of Lim [104] and Lami Dozo [99]. Xu [167] also proved a new fixed point theorem for *-nonexpansive multi-valued mappings.

Husain and Latif [74] quoted that "each *-nonexpansive mapping is weakly nonexpansive". Xu [167] corrected it and gave the following relationship between them.

Theorem 3.3.4. Let $T : X \to 2^X$ be a weakly nonexpansive multi-valued mapping, then it is nonexpansive. If T is compact-valued, then it is weakly nonexpansive if and only if it is nonexpansive.

Remark 3.3.2. Theorem 3.3.4 shows that the results of Husain and Tarafdar [75] and of Husain and Latif [74] on weakly nonexpansive multi-valued mappings are special cases of those of Lim [104] and Lami-Dozo [99].

*-nonexpansiveness is different from nonexpansive for multi-valued mappings, as shown by the following two examples due to Xu [167].

In what follows, $P_T(x)$ denotes the set

$$\{u_x \in T(x) : d(x, u_x) = \inf(d(x, u) : u \in T(x))\} \ \forall \ x \in X.$$

Example 3.3.3. (A *- nonexpansive mapping which is different from nonexpansive). Let $X = [0, \infty)$ and let $T : X \to \text{cpt}(X)$ be defined by

$$T(x) = [x, 2x] \ \forall \ x \in X.$$

Then $P_T(x) = \{x\}$ for every $x \in X$. This clearly implies that T is *-nonexpansive. However

$$H(T(x), T(y)) = H([x, 2x], [y, 2y]) = \max\{|x - y|, |2x - 2y|\} = 2|x - y|,$$

showing that T is not nonexpansive.

Example 3.3.4. (A nonexpansive mapping which is not *-nonexpansive). Let X be the triangle in the plane with vertices $O(0, 0), A(1, 0)$ and $B(0, 1)$. Let $T : X \to \text{cpt}(X)$ be given by

$$T(x, y) = \text{ the segment joining } (0, 1) \text{ and } (x, 0).$$

Then $P_T(x, y)$ is the only element $u_{(x,y)}$ in $T(x, y)$. It can be easily seen that

$$H(T(x_1, y_1), T(x_2, y_2)) = |x_1 - x_2| \leq d((x_1, y_1), (x_2, y_2))$$

for all $(x_i, y_i) \in X$ $(i = 1, 2)$, i.e., T is nonexpansive.
But for all $(x, y) \in X$ with $0 < x, y < 1$, we have

$$|u_{(x,y)} - u_{(1,0)}| > d((x, y), (1, 0)),$$

which shows that T is not *-nonexpansive.

Further, Xu [167] proved two new fixed point theorems for *-nonexpansive multi-valued mappings which cannot be implied by Lim [104] and Lami-Dozo [99] classical ones for nonexpansive multi-valued mappings.

Definition 3.3.3. Let C be a nonempty subset of a Banach space X and let $T : C \to 2^C$ be a multi-valued mapping. Then a single-valued map $f : C \to C$ is said to be a *selector* of T if $f(x) \in T(x)$ for each $x \in C$.

Theorem 3.3.5[167]. Let C be a weakly compact convex subset of a strictly convex Banach space X and let $T : C \to 2^C$ be a closed (not necessarily norm-compact) convex valued *-nonexpansive mapping. Then T possesses a nonexpansive selector. If, in addition, C has the f.p.p. for single-valued nonexpansive mappings, then T has a fixed point.

Theorem 3.3.6[167]. Let C be a weakly compact convex subset of a Banach space X satisfying Opial's condition and let $T : C \to 2^C$ be a norm-compact convex and *-nonexpansive mapping. Then T has a fixed point in C.

Remark 3.3.3. Theorem 3.3.6 removes the uniform convexity assumption on the space X of Husain and Latif [74], i.e., Theorem 3.3.3.

In this continuation, we record some fixed point theorems for *-nonexpansive mappings due to Shahzad and Lone [150] as follows:

Before stating the theorem we need the following definitions.

Definition 3.3.4. *Kuratowski* and *Hausdorff* measures of noncompactness of a nonempty bounded subset C of X are respectively defined as the numbers

(*i*) $\alpha(C) = \inf\{d > 0 : C$ can be covered by finitely many sets of diameter $\leq d\}$,

(*ii*) $\chi(C) = \inf\{d > 0 : C$ can be covered by finitely many balls of radius $\leq d\}$.

Definition 3.3.5. A multi-valued mapping $T : C \to 2^X$ is called $1 - \gamma$*-contractive* where $\gamma = \alpha(\cdot)$ or $\xi(\cdot)$ if, for each bounded subset K of C with $\gamma(C) > 0$, there holds the inequality

$$\gamma(T(C)) \leq \gamma(C).$$

Here $T(C) = \cup_{x \in C} Tx$.

Definition 3.3.6. The *separation measure of noncompactness* of a nonempty bounded subset C of X is defined by

$$\beta(C) = \sup\{\epsilon : \text{ there exists a sequence } \{x_n\} \text{ in } C \text{ such that } \operatorname{sep}(\{x_n\}) \geq \epsilon\}$$

where $\operatorname{sep}(\{x_n\}) = \inf\{\|x_n - x_m\| : n \neq m \geq \epsilon\}$.

Definition 3.3.7. Let X be a Banach space and $\phi = \alpha, \beta$ or χ. The *modulus of noncompact convexity* associated to ϕ is defined in the following way:

$$\Delta_{X,\phi}(\epsilon) = \inf\{1 - d(0, A) : A \subset B_X \text{ is convex and } \phi(A) \geq \epsilon\},$$

where B_X is the unit ball of X.

The characteristic of noncompact convexity of X associated with the measure of noncompactness ϕ is defined by

$$\epsilon_\phi(X) = \sup\{\epsilon \geq 0 : \Delta_{X,\phi}(\epsilon) = 0\}.$$

Theorem 3.3.7. Let C be a nonempty closed bounded and convex subset of a Banach space X such that $\epsilon_\alpha(X) < 1$ (Def. 3.3.7) and $T : C \to C(X)$ a *-nonexpansive, $1 - \chi$-contractive mapping. If T satisfies

$$T(x) \subset I_C(x) \ \forall\, x \in C,$$

then T has a fixed point.

From above result following corollary can be deduced which extends Theorem 3.3.6 to nonself multi-valued mappings and to spaces satisfying the nonstrict Opial condition.

Corollary 3.3.4. Let C be a nonempty closed bounded and convex subset of a Banach space X such that $\epsilon_\alpha(X) < 1$ satisfying the nonstrict Opial condition and $T : C \to C(X)$ a *-nonexpansive mapping. If T satisfies

$$T(x) \subset I_C(x) \ \forall\, x \in C,$$

then T has a fixed point.

Benavides and Ramirez [8] proved fixed point theorem for a multi-valued nonexpansive and $1-\gamma$-contractive mappings in the framework of a Banach space whose characteristic of noncompact convexity associated to the separation measure of noncompactness is less than 1. Fixed point theorems for multi-valued nonexpansive self mappings proved in [8] are more general than the earlier results.

Theorem 3.3.8. Let C be a nonempty closed bounded and convex subset of a Banach space X such that $\epsilon_\beta(X) < 1$ and $T : C \to C(C)$ be a nonexpansive and $1 - \chi$-contractive nonexpansive mapping, then T has a fixed point.

Remark 3.3.4. Theorem 3.3.8 does not hold if nonexpansiveness assumption is removed. Indeed, if B is the closed unit ball of l_2 and $T : B \to B$ is defined by

$$T(x) = T(x_1, x_2, ...) = (\sqrt{1 - \|x\|^2}, x_1, x_2, ...),$$

then T is $1-\chi$-contractive without a fixed point.

Theorem 3.3.9. If X is either a separable or reflexive Banach space satisfying the nonstrict Opial's condition, C is a nonempty weakly compact subset of X and $T : C \to C(C)$ is a nonexpansive mapping, then T is $1-\chi$-contractive.

In view of the above result following corollary can be deduced immediately.

Corollary 3.3.5. Let X be a Banach space with $\epsilon_\beta(X) < 1$ which satisfies the nonstrict Opial's condition. Suppose C is a nonempty weakly compact convex subset of X and $T : C \to C(C)$ is a nonexpansive mapping, then T has a fixed point.

Theorem 3.3.10. Let X be a Banach space with $\epsilon_\chi(X) < 1$ which satisfies the nonstrict Opial condition. Suppose C is a nonempty weakly compact convex subset of X and $T : C \to C(C)$ is a nonexpansive mapping, then T has a fixed point.

Benavides and Ramirez [8] remarked that the assumption of nonexpansiveness cannot be avoided, Shahzad and Lone [150] proved fixed point result for multi-valued mappings which are not necessarily nonexpansive. To establish this, Shahzad and Lone [150] defined a new class of multi-valued mappings which includes nonexpansive mappings.

Definition 3.3.8. Let C be a nonempty weakly compact convex subset of a Banach space X and $T : C \to C(X)$ a continuous mapping. The map T is called *subsequentially limit-contractive (SL)* if for every asymptotically regular sequence $\{x_n\}$ in C,

$$\limsup_n H(Tx_n, Tx) \leq \limsup_n \|x_n - x\|,$$

for all $x \in A_C(\{x_n\})$.

Note that if C is a nonempty closed convex subset of a uniformly convex Banach space X and $\{x_n\}$ is bounded, then $A_C(\{x_n\})$ has a unique asymptotic center, say x_0, and so in Definition 3.3.8., we have

$$\limsup_n H(Tx_n, Tx_0) \leq \limsup_n \|x_n - x_0\|.$$

Every nonexpansive mapping is an SL mapping. Several examples of mappings can be constructed which are SL but not nonexpansive. For example, we give following one:

Example 3.3.5. Let $C = [0, 3/5]$ with the usual norm and consider the mapping $Tx = x^2$. It is easy to see that T is an SL mapping but not nonexpansive. Moreover, T is $1-\chi$-contractive and has a fixed point.

Following theorem due to Shahzad and Lone [150] contains Theorem 3.3.8 as a special case.

Theorem 3.3.11. Let C be a nonempty closed bounded and convex subset of a Banach space X such that $\epsilon_\alpha(X) < 1$ and $T : C \to C(X)$ a continuous, SL and $1 - \chi$-contractive mapping. If T satisfies

$$T(x) \subset I_C(x), \ \forall\, x \in C,$$

then T has a fixed point.

Corollary 3.3.6. Let C be a nonempty closed bounded and convex subset of a Banach space X satisfying Opial's condition such that $\epsilon_\alpha(X) < 1$ and $T : C \to C(X)$ a nonexpansive mapping. If T satisfies

$$T(x) \subset I_C(x), \ \forall\, x \in C,$$

then T has a fixed point.

§ 3.4. Structure of fixed point set

Throughout this section C is assumed to represents a closed convex subset of a Banach space X and $F(T)$ the set of fixed points of a mapping $T : X \to 2^X$ to be nonempty.

Definition 3.4.1. A mapping $T : X \to CB(X)$ is *strictly nonexpansive* if

$$H(Tx, Ty) < \|x - y\| \text{ for any } x, y \in X \text{ and } x \neq y.$$

If T is a single-valued mapping, then the following properties are true.

(a) If T is strictly nonexpansive, then $F(T)$ is a singleton.

(b) If T is nonexpansive and the norm of the Banach space X is strictly convex, then $F(T)$ is convex.

Statement (a) is no longer true for multi-valued mappings. For example, let C be a set containing more than two points, then the set of fixed points of the mapping $T : C \to 2^C$ such that $T(x) = C$ for any $x \in C$, is C itself which is not a singleton.

Following example shows that the statement (b) is also not true for multi-valued mappings such that the image of each point is a nonempty convex set.

Example 3.4.1. Let $C = [0, 1] \times [0, 1]$ be a subset of $\Re^2$ with the usual norm and $T : C \to C(X)$ be a nonexpansive mapping defined by

$$T((x_1, x_2)) = \text{the triangle with vertices } (0, 0), (x_1, 0) \text{ and } (0, x_2).$$

Note that $T((x_1, x_2))$ is a degenerate triangle if $x_1x_2 = 0$ and the norm in $\Re^2$ is strictly convex. But the set $F(T)$ of fixed points of T is

$$F(T) = \{(x_1, x_2) : \ (x_1, x_2) \in C \text{ and } x_1x_2 = 0\}$$

which is not convex.

For a multi-valued mapping T, one have several choices for values of T, e.g. $T(x) \in K(X), T(x) \in \mathrm{cpt}(X)$ or $T(x) \in C(X)$; among them, $T(x) \in C(X)$ is the strongest assumption. For example, let C be a compact convex subset of X and let $T : X \to \mathrm{cpt}(X)$ be an upper semicontinuous mapping such that $T(x) \subset C$ for any $x \in C$, then T does not always have a fixed point. But if we simply change T as a mapping into $C(X)$ instead of into $\mathrm{cpt}(X)$, then T has a fixed point. In above example, although we have imposed the strongest condition on the values of T, i.e., $T(x) \in C(X)$, that condition does not force T to satisfy (b). However, the following theorem due to Ko [96] gives the sufficient condition for $F(T)$ to be convex. But first we state following lemma which will be used in the next theorem.

Lemma 3.4.1. Let $T : X \to 2^X$, define

$$H_r = \{x \in X : d(x, Tx) \leq r\},$$

where $r \geq 0$. If $I - T$ is a semiconvex mapping on X, then H_r is convex.

Theorem 3.4.1[96]. Let $T : C \to 2^C$ be a mapping such that $I - T$ is a semiconvex mapping on C. Then $F(T)$ is convex.

Proof.
If $I - T$ is semiconvex on C, then above lemma shows that the set

$$H_r = \{x \in X : d(x, Tx) \leq r\}$$

is convex. Hence $F(T) = H_0$ is convex.

* * * * *

CHAPTER 4

ON ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

§ 4.1. Introduction

As discussed in Chapter 2^{nd}, Browder [19], Göhde [66] and Kirk [87] simultaneously but independently proved that every nonexpansive self mapping of a nonempty closed bounded convex subset of a uniformly convex Banach space always admit a fixed point (see Theorem 2.2.3). Here it may be pointed out that the same is not true if we allow the Lipschitz constant k to take values greater then 1 in case the underlying space is a Hilbert space.

With a view to widen the class of nonexpansive mappings, Goebel and Kirk [61, 1972] noticed that there are classes of mappings which lie between the nonexpansive mappings and those with Lipschitz constant $k > 1$ for which fixed point theorems do exist; one such class of mappings was termed as 'asymptotically nonexpansive' by Goebel and Kirk [61] as mentioned earlier. These are mappings $T : C \to C$ defined on a nonempty subset C of a Banach space X having the property that T^n has Lipschitz constant k_n with $k_n \to 1$ as $n \to \infty$. The class of asymptotically nonexpansive mappings is wider than the class of nonexpansive mappings. The following example illustrates the situation better.

Example 4.1.1[61]. Let B denote the unit ball in the Hilbert space l^2 and let T be defined as follows:

$$T : (x_1, x_2, x_3, \cdots) \to (0, x_1^2, r_2x_2, r_3x_3, \cdots)$$

where r_i is a sequence of numbers such that $0 < r_i < 1$ and $\Pi_{i=2}^{\infty} r_i = \frac{1}{2}$. Then T is Lipschitzian and

$$\|Tx - Ty\| \leq 2\|x - y\|, \quad x, y \in B,$$

and moreover,

$$\|Tx - Ty\| \leq 2\Pi_{j=2}^{i} r_j \|x - y\| \ \text{ for } i = 2, 3, \cdots .$$

Thus

$$\lim_{i\to\infty} k_i = \lim_{i\to\infty} 2\Pi_{j=2}^{i} r_j = 1.$$

Clearly, the mapping T is not nonexpansive.

Definition 4.1.1. A Lipschitzian mapping T from a nonempty (and generally, closed bounded convex) subset C of linear space X satisfying

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \ \forall \ x, y \in C \ \text{ and } \ n \in N$$

is called

(a) *uniformly k-Lipschitzian* if $k_n = k$ for all $n \geq 1$

(b) *nonexpansive* if $k_n = 1$, for all $n \geq 1$

(c) *asymptotically nonexpansive* if $k_n \geq 1$, $\forall\, n \geq 1$, and $\lim_{n\to\infty} k_n = 1$.

Fixed point theory for nonexpansive mappings has been studied extensively and the existing literature on asymptotically nonexpansive mappings is very extensive. In [61], Goebel and Kirk proved the natural generalization of Browder-Göhde-Kirk theorem(Theorem 2.2.3) using the notion of asymptotically nonexpansive mapping. Later in 1973, the same authors [62] proved that the result remains valid for the broader class of uniformly k-Lipschitzian mappings with $k < \gamma$, where γ is sufficiently near to one. This was extended to mapping of asymptotically nonexpansive type by Kirk [91]. The result of Goebel and Kirk [61] has been further generalized to a k-uniformly rotund Banach space for any integer $k \geq 1$ by Xu [165] and more generally to a nearly uniformly convex Banach space by Xu [166]. More recently these results have been extended to wider classes of spaces, for example see [31, 50, 86, 106, 108]. In particular Lim and Xu [106] and Kim and Xu [86] have demonstrated the existence of fixed points for asymptotically nonexpansive mappings in Banach spaces with uniform normal structure, see also [33] for some related results. However, great deal of work has been done on the asymptotic aspect of the fixed point theory but many natural questions remains open out of which following are worthy of recording.

Question 4.1.1. Whether normal structure implies the existence of fixed points for mappings of asymptotically nonexpansive type.

Question 4.1.2. Let C be a closed bounded and convex subset of a Banach space X and given a mapping $T : C \to X$ with Lipschitz constant $k_1 > 1$, it is natural to ask whether one can say anything about the sequence $\{k_n\}$ where

$$k_n = \sup\left\{\frac{\|T^n x - T^n y\|}{\|x - y\|} : \quad x, y \in C; x \neq y\right\}.$$

The answer seems to be negative.

Question 4.1.3. It is well known that if B is a unit ball in l_∞ then every nonexpansive self mapping of B has a fixed point. But the question of whether every asymptotically nonexpansive self mapping of B has a fixed point remains open.

§ 4.2. Some existence theorems

Goebel and Kirk [61] proved the following result which is a generalization of a result due to Browder [19] for asymptotically nonexpansive mappings.

Theorem 4.2.1[61]. Let C be a nonempty closed bounded and convex subset of a uniformly convex Banach space X. Then $T : C \to C$ has a fixed point if T is asymptotically nonexpansive.

Proof.

Let $B(x, r)$ denote the ball centered at $x \in C$, having radius r. Let $y \in C$ be fixed and let $\Re_y$ be the set consisting of those numbers ρ for which there exists an integer k such that

$$C \cap \left(\bigcap_{i=k}^{\infty} B(T^i y, \rho) \right) \neq \emptyset.$$

If d is the diameter of C then $d \in \Re_y$, so $\Re_y \neq \emptyset$. Let $\rho_0 = $ g.l.b. $\Re_y$ and define $C_\epsilon = \bigcup_{k=1}^{\infty} (\bigcap_{i=k}^{\infty} B(T^i y, \rho_0 + \epsilon))$ for each $\epsilon > 0$. Thus the sets $C_\epsilon \cap C$ are nonempty and convex for each $\epsilon > 0$, so reflexivity of X implies that

$$K = \bigcap_{\epsilon > 0} (\overline{C_\epsilon} \cap C) \neq \emptyset.$$

Note that for $x \in C$ and $\eta > 0$ there exists an integer N such that if $i \geq N$, $\|x - T^i y\| \leq \rho_0 + \eta$.

Now let $x \in K$ and suppose the sequence $\{T^n x\}$ does not converge to x (i.e., $Tx \neq x$). Then there exists $\epsilon > 0$ and a subsequence $\{T^{n_i} x\}$ of $\{T^n x\}$ such that $\|T^{n_i} x - x\| \geq \epsilon$, $i = 1, 2, \cdots$. For $m > n$,

$$\|T^n x - T^m x\| \leq k_n \|x - T^{m-n} x\|,$$

where k_n is the Lipschitz constant for T^n obtained from the definition of asymptotically nonexpansiveness. Assume $\rho_0 > 0$ and choose $\alpha > 0$ so that $(1 - \delta(\epsilon/(\rho_0 + \alpha)))(\rho_0 + \alpha) < \rho_0$. Select n so that $\|x - T^n x\| \geq \epsilon$ and also so that $k_n(\rho_0 + \alpha/2) \leq \rho_0 + \alpha$. If $N \geq n$ is sufficiently large, then $m > N$ implies

$$\|x - T^{m-n} y\| \leq \rho_0 + \alpha/2,$$

and one gets

$$\|T^n x - T^m y\| \leq k_n \|x - T^{m-n} y\| \leq \rho_0 + \alpha,$$

$$\|x - T^m y\| \leq \rho_0 + \alpha.$$

Thus by uniform convexity of X, for $m > N$,

$$\left\| \frac{x + T^n x}{2} - T^m y \right\| \leq \left(1 - \delta \left(\frac{\epsilon}{\rho_0 + \alpha} \right) \right) (\rho_0 + \alpha) < \rho_0,$$

which is a contradiction to the definition of ρ_0. Hence $\rho_0 = 0$ or $Tx = x$. But $\rho_0 = 0$ implies $\{T^n y\}$ is a Cauchy sequence yielding $T^n y \to x = Tx$ as $n \to \infty$. Therefore the set C consists of a single point which is fixed under T.

In 1998, Kirk et al. [95] showed that there is a close relationship between the fixed point property (f.p.p.) for nonexpansive mappings and the approximate fixed point property (a.f.p.p.) for asymptotically nonexpansive mappings and used the approach of ultrapower to give a very quick new proof of Theorem 4.2.1.

Definition 4.2.1. Let C be a nonempty set. A *filter* $\mathcal{F}$ in C is a collection of nonempty subsets of C satisfying

(*a*) $A, B \in \mathcal{F} \implies A \cap B \in \mathcal{F}$,

(*b*) $A \in \mathcal{F}$ and $A \subset B \implies B \in \mathcal{F}$.

The family of all filters on C, ordered by set inclusion, satisfies the principle that each of its linearly ordered chain has an upper bound (the union of all filters in the family). Therefore, by Zorn's lemma, every such filter is contained in a maximal filter (one not contained in any larger filter). Any filter maximal in this sense is called an *ultrafilter.* The simplest ultrafilters are those generated by a single element $x \in C$ (i.e., consisting of all subsets of C which contain x). These ultrafilters are called *trivial* and all other ultrafilters are said to be *free ultrafilters.*

Definition 4.2.2. Let X be a Banach space and let U be a nontrivial ultrafilter over the set of natural numbers N. Let

$$l_\infty(X) = \{x = \{x_n\} \in X : \sup_{1 \le i < \infty} \|x_i\| < \infty\},$$

and let

$$N = \{x = \{x_n\} \in l_\infty(X) : \lim_U \|x_n\| = 0\}.$$

The *Banach space ultrapower* $\overline{X}$ *of* X *over* U is the quotient space $l_\infty(X)/N$. Thus the elements of $\overline{X}$ are equivalence classes of the bounded sequence $\{x_n\} \subset X$, where one agrees that two such sequences $\{x_n\}$ and $\{y_n\}$ are equivalent if

$$\lim_U \|x_n - y_n\| = 0.$$

The norm $\|.\|_U$ in $\overline{X}$ is given by defining for $\overline{x} = [\{x_n\}] \in \overline{X}$,

$$\|\overline{x}\|_U = \lim_U \|x_n\|.$$

Now let $C \subseteq X$ and suppose $T : C \to C$. Letting

$$\overline{C} = \{\overline{x} = [\{x_n\}] \in \overline{X} : x_n \in C, \ \forall\, n\}.$$

There is a canonical way to extend T to a mapping $\overline{T} : \overline{C} \to \overline{C}$ by setting for $\overline{x} = [\{x_n\}] \in \overline{C}$,

$$\overline{T}(\overline{x}) = [\{T(x_n)\}].$$

Here, one may note that $\overline{C}$ is a closed bounded and convex subset of $\overline{X}$ and $\overline{T}$ is nonexpansive provided T is nonexpansive as well. Moreover, if C is bounded and T is nonexpansive then it is well known that there always exist sequence $\{x_n\} \subset C$ such that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

From this it follows that $\overline{T}\overline{x} = \overline{x}$, where $\overline{x} = [\{x_n\}]$, i.e., $F(\overline{T}) \neq \emptyset$.

It is assumed that T is asymptotically nonexpansive then $\overline{T}$ will be asymptotically nonexpansive but this is not true for nonexpansive mappings.

A Banach space X has *super-f.p.p. if* $\overline{X}$ has the f.p.p. for nonexpansive mappings, where $\overline{X}$ is any ultrapower of X over some nontrivial ultrafilter in N. It is known that such a space $\overline{X}$ is necessarily reflexive (hence superreflexive) as particularly speaking, it has the f.p.p. for isometries.

Theorem 4.2.2[95]. If a Banach space X has the super-f.p.p. for nonexpansive mappings, then X has the a.f.p.p. for asymptotically nonexpansive mappings.

Using this approach, Kirk et al. [95] gave the following proof of Theorem 4.2.1.

Theorem 4.2.3[95]. Let C be a closed bounded and convex subset of a uniformly convex Banach space X and suppose $T : C \to C$ is asymptotically nonexpansive. Then T has a fixed point.

Proof.

Let $\overline{T}$ and $\overline{C}$ be defined as above. Since the modulus of convexity(Def. 1.2.4) is a super property, $\overline{X}$ is uniformly convex and in particular X has the super f.p.p..

Let $\dot{C}$ be a closed convex subset of $\overline{C}$ defined as follows:

$$\dot{C} = \{\dot{x} = [\{x_n\}] : x_n = x \in C\}.$$

Thus by the Theorem 4.2.2, $\overline{T}$ has a fixed point $\overline{x} \in \overline{C}$. Note that if $\overline{x} = \dot{x} \in \dot{C}$ then $\|\dot{x} - \overline{T}(\dot{x})\|_U = \lim_U \|T(x) - x\| = 0$ and it follows that $T(x) = x$. On the other hand, if $\overline{x} \notin \dot{C}$ then there is a unique point $\dot{z} \in \dot{C}$ such that $\|\overline{x} - \dot{z}\|_U = D(\overline{x}, \dot{C})$ and since $\overline{X}$ is uniformly convex $\lim_n \|\overline{T}^n(\dot{z}) - \dot{z}\|_U = 0$, from which $\overline{T}(\dot{z}) = \dot{z}$. Therefore $T(z) = z$ and in either case theorem is proved.

Definition 4.2.3. Let X be a Banach space. Then the *normal structure coefficient of* X(denoted by $N(X)$) is defined by

$$N(X) = \inf\left\{\frac{\delta(C)}{r_C(C)}\right\},$$

where infimum is taken over all closed bounded and convex subsets C of X with more than one point, $\delta(C) = \sup\{\|x - y\| : x, y \in C\}$ is the diameter of C and

$r_C(C) = \inf\{\sup\{\|x - y\| : y \in C\} : x \in C\}$ is the self Chebyshev radius of C. If $N(X) > 1$ then X has uniform normal structure.

A deep result of Casini and Maluta [33] is following:

Theorem 4.2.4[33]. If $k < \sqrt{N(X)}$, where $N(X)$ is the normal structure coefficient of X, then a uniformly k-Lipschitzian mapping $T : C \to C$ has a fixed point.

Though an asymptotically nonexpansive mapping T is eventually uniformly Lipschitzian. Theorem 4.2.4 does not apply to an asymptotically nonexpansive mapping. Next is a result due to Lim and Xu [106] which is a slight generalization of Theorem 4.2.4.

Theorem 4.2.5[106]. Suppose C be a nonempty bounded subset of a Banach space X with uniform normal structure and $T : C \to C$ is a uniformly k-Lipschitzian mapping with $k < \sqrt{N(X)}$. Suppose that there also exists a nonempty closed bounded and convex subset K of C along with the following property:

$$x \in K \implies \omega_w(x) \subset K, \tag{4.2.1}$$

where $\omega_w(x)$ is the weak ω-limit set of T at x, i.e., the set

$$\{y \in X : y = \text{weak}\lim_j T^{n_j}x \quad \text{for some} \quad n_j \to \infty\}.$$

Then T has a fixed point in K.

Remark 4.2.1. Note that if C itself is a nonempty weakly compact and convex subset of X, then Theorem 4.2.5 reduces to Theorem 4.2.4.

A direct consequence of Theorem 4.2.5. is the following result.

Theorem 4.2.6[106]. Let T be an asymptotically nonexpansive self mapping of a nonempty bounded subset C of a uniformly smooth Banach space X. If there exists a nonempty closed bounded and convex subset K of C satisfying (4.2.1), then T has a fixed point in K.

In continuation, Kim and Xu [86, 2000] established the existence of fixed point of asymptotically nonexpansive mappings in space having uniform normal structure.

Theorem 4.2.7[86]. If T be an asymptotically nonexpansive self mapping of a closed bounded and convex subset of a Banach space X equipped with uniform normal structure, then T has a fixed point.

Since uniform smoothness implies normal structure, one have the following result which was implicitly used in Theorem 4.2.5.

Theorem 4.2.8 [86]. Every asymptotically nonexpansive mapping T defined on a closed bounded and convex subset of a uniformly smooth Banach space X has a fixed point.

Since uniformly convex Banach space is uniformly smooth, above result generalizes Theorem 4.2.1. Kim and Xu [86] raised a question about the existence of Theorem 4.2.7 for mappings of asymptotically nonexpansive type. Li and Sims [102, 2002] presented answer to this question. Their results [102] were proved with an endeavour to extend the results in [86, 106] to the mappings of asymptotically nonexpansive type.

Definition 4.2.4[91]. Let C be a nonempty subset of a Banach space X. Then a mapping $T : C \to C$ is said to be of *asymptotically nonexpansive type* if for each $x \in C$

$$\lim_{i\to\infty} \sup\{\sup_{y\in C}[\|T^i(x) - T^i(y)\| - \|x - y\|]\} \leq 0.$$

Theorem 4.2.9[102]. Let T be a continuous self mapping of asymptotically nonexpansive type defined on a nonempty bounded subset of a Banach space X equipped with a uniform normal structure. If there exists a nonempty closed convex subset K of C satisfying (4.2.1), then T has a fixed point in K.

Since an asymptotically nonexpansive mapping is of asymptotically nonexpansive type. Theorem 4.2.6 can be derived as a corollary to Theorem 4.2.9.

Corollary 4.2.1[102]. Let C and X be as in Theorem 4.2.9 and let $T : C \to C$ be an asymptotically nonexpansive mapping. Suppose there exists a nonempty closed bounded subset K of C satisfying (4.2.1). Then T has a fixed point.

Following corollary can be considered as a slight generalization of Theorem 4.2.1.

Corollary 4.2.2[102]. Let X be a Banach space with uniform normal structure, let C be a closed bounded and convex subset of X and suppose $T : C \to C$ is a continuous mapping of asymptotically nonexpansive type. Then T has a fixed point.

§ 4.3. Iterations in Banach and Hilbert spaces

After the introduction of the class of asymptotically nonexpansive mappings, several researchers utilized the process of iterative construction of a fixed point to prove results on asymptotically nonexpansive mapping as the weak limit of the sequence of iterates, assuming that T is (weakly) asymptotically regular (see for example [11, 67, 123, 141]).

The study of iterative construction for fixed points of asymptotically nonexpansive mappings was initiated in 1978. Bose [11] first proved that if C is a closed bounded and convex subset of a uniformly convex Banach space X which satisfies Opial's condition and $T : C \to C$ is an asymptotically nonexpansive, then $\{T^n x\}$ converges weakly to a fixed point of T provided T is asymptotically regular at x_0. This conclusion is still valid [123, 166] if Opial's condition of X is replaced by the condition that X has Frèchet differentiable norm. Furthermore, in both cases, asymptotic regularity of T at x can be weakened to weak asymptotic regularity of T at x, i.e.,

$$\mathrm{w} - \lim_{n\to\infty} \|T^n x - T^{n+1}x\| = 0.$$

In 1991, Schu [144] established that under appropriate conditions, the modification $x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n$ of the usual Mann iteration converges strongly to some fixed point of T provided T is completely continuous and asymptotically nonexpansive. Since then, Schu's [144] iterations process has been widely used to approximate fixed points of asymptotically nonexpansive self mappings in Hilbert spaces or Banach spaces.

Theorem 4.3.1[144]. Let C be a nonempty closed bounded and convex subset of a Hilbert space X. $T : C \to C$ be an asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$, $\sum\limits_{n=1}^{\infty}(k_n^2 - 1) < \infty$, $\{\alpha_n\} \subset [0,1], \epsilon \leq \alpha_n \leq 1-\epsilon$ for all $n \in N$ and some $\epsilon > 0$. Let $x_1 \in C$ define

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n \;\forall\; n \in N \tag{4.3.1}$$

then $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$.

Theorem 4.3.2[144]. Let C be a nonempty closed bounded and convex subset of a Hilbert space X. $T : C \to C$ be a completely continuous and asymptotically nonexpansive with sequence $\{k_n\} \subset [1, \infty)$, $\sum\limits_{n=1}^{\infty}(k_n^2 - 1) < \infty$ $\{\alpha_n\} \subset [0,1], \epsilon \leq \alpha_n \leq 1-\epsilon \;\forall\; n \in N$ and some $\epsilon > 0$. Let $\{x_n\}$ be the usual Mann iteration process defined by (4.3.1). Then $\{x_n\}$ converges strongly to some fixed point of T.

Following result is due to Schu [145].

Theorem 4.3.3[145]. Let C be a nonempty closed bounded and convex subset of a uniformly convex Banach space X satisfying Opial's condition and $T : C \to C$ be an asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$, $\sum\limits_{n=1}^{\infty}(k_n - 1) < \infty$. Suppose that $x_1 \in C$ and $\{\alpha_n\} \subset [0,1]$ is bounded away. Then the sequence $\{x_n\}$ given by (4.3.1) converges weakly to some fixed point of T.

Following theorem due to Schu [145] is a generalization of Theorem 4.3.2, when X is restricted to be a Hilbert space.

Theorem 4.3.4[145]. Let C be a nonempty closed bounded and convex subset of a uniformly convex Banach space X. Let $T : C \to C$ be an asymptotically nonexpansive with sequence $\{k_n\} \subset [1, \infty)$ for which $\sum\limits_{n=1}^{\infty}(k_n - 1) < \infty$. Suppose that $x_1 \in C$ and $\{\alpha_n\} \subset [0, 1]$ is bounded away. If T^m is compact for some $m \in N$, then the sequence $\{x_n\}$ described by (4.3.1) converges strongly to some fixed point of T.

Unfortunately, Theorem 4.3.3 does not apply to the L^p space if $p \neq 2$ since none of these spaces satisfy Opial's condition. Tan and Xu [160] showed that Theorem 4.3.3 remains true if the assumption that X satisfies Opial's condition is replaced by the one that X has a Frèchet differentiable norm. Tan and Xu [160] result applies to the L^p spaces for $1 < p < \infty$ since each of these spaces is uniformly convex and uniformly smooth.

Theorem 4.3.5[160]. Let C be a closed bounded and convex subset of a uniformly convex Banach space X with a Frèchet differentiable norm. If $T : C \to C$ be an asymptotically nonexpansive mapping such that $\sum\limits_{n}(k_n - 1)$ converges, then for each $x_1 \in C$, the sequence $\{x_n\}$ defined by (4.3.1) (with $\{\alpha_n\}$ a sequence of real numbers bounded away from 0 and 1) converges weakly to a fixed point of T.

Remark 4.3.1. It is not known that Theorem 4.3.5 remains valid if k_n is allowed to approach 1 slowly enough so that $\sum\limits_{n}(k_n - 1)$ diverges.

Schu [144] established the convergence of the Mann iterates of a completely continuous asymptotically nonexpansive mapping on a Hilbert space. In 1994, Rhoades [139] extended Theorem 4.3.2 to uniformly convex Banach spaces.

Theorem 4.3.6[139]. Let C be a nonempty closed bounded and convex subset of a uniformly convex Banach space X. T be an asymptotically nonexpansive self mapping of C with $\{k_n\} \geq 1$, $\sum\limits_{n=1}^{\infty}(k_n^r - 1) < \infty$, for some $r > 1$, $\epsilon \leq 1 - \alpha_n \leq 1 - \epsilon$ for all n and some $\epsilon > 0$. If for some $x_1 \in C$, $\{x_n\}$ is described by (4.3.1), then $\lim \|x_n - Tx_n\| = 0$.

Theorem 4.3.7. Let C be a nonempty closed bounded convex subset of a uniformly convex Banach space. Let T be a completely continuous asymptotically nonexpansive self mapping of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum\limits_{n=1}^{\infty}(k_n^r - 1) < \infty$, $r = \max\{2, p\}$ where $p > 1$, $\epsilon \leq \alpha_n \leq 1 - \epsilon$ for all n and some $\epsilon > 0$. If some $x_1 \in C$, the sequence $\{x_n\}$ is described by (4.3.1), then $\{x_n\}$ converges strongly to some fixed point of T.

Remark 4.3.2. Theorem 4.3.2 is a special case of Theorem 4.3.7 for $p = 2$.

Following is a comparison theorem for an Ishikawa type iteration which is given by Rhoades [139].

Theorem 4.3.8[139]. Let C be a nonempty closed bounded and convex subset of uniformly convex Banach space X. Let T be a completely continuous asymptotically nonexpansive self mapping of C with $\{k_n\}$ satisfying $k_n \geq 1$, $\sum_{n=1}^{\infty}(k_n^r - 1) < \infty, r = \max\{2, p\}, p > 1$. Define $\{\alpha_n\}, \{\beta_n\}$ to satisfy $\epsilon \leq (1-\alpha_n), (1-\beta_n) \leq 1-\epsilon$ for all n and some $\epsilon > 0$. Define

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n, \ \forall\, n \geq 0 \\ \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n. \end{cases}$$

Then $\{x_n\}$ converges strongly to a fixed point of T.

Huang [72] extended Theorems 4.3.7 and 4.3.8 of [139] to further general cases.

Definition 4.3.1[72]. Let C be a subset of a Banach space X. The *Ishikawa iteration process* $\{x_n\}$ *with errors* (with $x_1 \in C$) is defined as follows

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n + u_n, \qquad n \in N \\ \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n + v_n, \end{cases} \tag{4.3.2}$$

where $\{u_n\}$ and $\{v_n\}$ are two sequences in C satisfying $\sum_{n=1}^{\infty} \|u_n\| < \infty$, $\sum_{n=1}^{\infty} \|v_n\| < \infty$, and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences of real numbers in $[0,1]$ satisfying suitable conditions.

Ishikawa iteration process with errors is a generalized case of the Ishikawa iteration process, while for all $n \in N$ with $\beta_n = 0$, it reduces to the Mann iteration process with errors which is a generalized case of the Mann iteration process.

Theorem 4.3.9[72]. Let C be a nonempty closed bounded and convex subset of a uniformly convex real Banach space X and $T : C \to C$ be a completely continuous asymptotically nonexpansive mapping with $\{k_n\} \subset [1, \infty)$, $\sum_{n=1}^{\infty}(k_n^r - 1) < \infty$ for some $r > 1$. Suppose that $\{\alpha_n\}$ is a sequence in $[0,1]$ satisfying $0 < a_1 \leq \alpha_n \leq 1 - a_2 < 1$ for all $n \geq 0$, where $a_1, a_2 \in (0,1)$ are some constants. For any $x_1 \in C$, define the following Mann iterative sequence with errors $\{x_n\}$ in C by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n + u_n, \qquad n \geq 0$$

where $\{u_n\}$ is a sequence in C satisfying $\sum_{n=1}^{\infty} \|u_n\| < \infty$. Then the sequence $\{x_n\}$ converges strongly to some fixed point of T.

Theorem 4.3.10[72]. Let C be a nonempty closed bounded and convex subset of a uniformly convex real Banach space X and $T : C \to C$ be a completely continuous asymptotically nonexpansive mapping with $\{k_n\} \subset [1, \infty)$, $\sum_{n=1}^{\infty}(k_n^r - 1) < \infty$ for some $r > 1$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences satisfying $0 < a_1 \leq \alpha_n \leq 1 - a_2 < 1$, $0 \leq \beta_n < 1$ for all $n \geq 0$, $\lim_{n\to\infty} \sup \beta_n \leq b < 1$ for all $n \geq 0$, where $a_1, a_2 \in (0,1)$ and $b \in [0,1)$ are some constants. Define Ishikawa iterative sequence with errors $\{x_n\}$ in C by (4.3.2). Then the sequence $\{x_n\}$ converges strongly to some fixed point of T.

Remark 4.3.3. If $u_n = v_n = 0$ for all $n \in N$, with $a_1 = a_2 = \epsilon$ (for some constant $\epsilon > 0$), then Theorem 4.3.9 reduces to Theorem 4.3.7

If $u_n = v_n = 0$ for all $n \in N$, $a_1 = a_2 = \epsilon$, '$0 \leq \beta_n < 1$', and '$\lim_{n\to\infty} \sup \beta_n \leq b < 1$' is replaced by '$\epsilon \leq \beta_n \leq 1 - \epsilon$' for some constant $\epsilon > 0$, then Theorem 4.3.10 reduces to Theorem 4.3.8. In fact, '$0 \leq \beta_n < 1$, $\lim_{n\to\infty} \sup \beta_n \leq b < 1$' can be deduced from '$\epsilon \leq \beta_n \leq 1 - \epsilon$' for some constants $\epsilon > 0$. It yields that Theorem 4.3.8 is a direct corollary of Theorem 4.3.10.

Remark 4.3.4. Theorem 4.3.8 cannot reduce to Theorem 4.3.7 since the condition $\epsilon \leq \beta_n \leq 1 - \epsilon \ \ \forall\, n \in N$ and some constant $\epsilon > 0$. By substituting $\beta_n = 0$, for all $n \in N$ Theorem 4.3.10, which generalizes Theorem 4.3.8, can reduce to Theorem 4.3.9 in the case of the Mann iteration with errors, which actually generalizes Theorem 4.3.7.

In recent years, one step and two step iterative schemes have been studied extensively to solve the nonlinear operator equations as well as variational inequalities in Hilbert and Banach spaces. Inspired and motivated by this fact, Xu and Noor [164] in 2002 suggested and analyzed a new class of three step iterative schemes for solving the nonlinear equation $Tx = x$ for asymptotically nonexpansive mappings in Banach spaces. This new iterative scheme includes Ishikawa type and Mann type iterations as special case.

Algorithm 4.3.1. Let C be a nonempty subset of a normed space X and let $T : C \to C$ be a mapping. For a given $x_1 \in C$, sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ can be computed by the iterative schemes

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n y_n, \ \forall\, n \geq 0 \\ y_n = (1 - \beta_n)x_n + \beta_n T^n z_n \\ z_n = (1 - \gamma_n)x_n + \gamma_n T^n x_n \end{cases} \tag{4.3.3}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$.

If $\gamma_n = 0$, then Algorithm 4.3.1 reduces to

Algorithm 4.3.2. Let C be a nonempty subset of a normed space X and let $T : C \to C$ be a mapping. For a given $x_1 \in C$, sequences $\{x_n\}$ and $\{y_n\}$ can be computed by the iterative schemes

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n, \ \forall\, n \geq 0 \\ \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n \end{cases} \tag{4.3.4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$. Notice that Algorithm 4.3.2 is Ishikawa iterative process.

For $\beta_n = 0$ and $\gamma_n = 0$, Algorithm 4.3.1 reduces to

Algorithm 4.3.3. Let C be a nonempty subset of a normed space X and let $T : C \to C$ be a mapping. For a given $x_1 \in C$, sequence $\{x_n\}$ can be computed by the iterative scheme

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n \qquad \forall\, n \geq 0 \tag{4.3.5}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. Algorithm 4.3.3 is Mann iterative process.

Theorem 4.3.11[164]. Let C be a nonempty closed bounded and convex subset of a uniformly convex Banach space X. Let T be a completely continuous asymptotically nonexpansive self mapping of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum\limits_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $[0,1]$ satisfying

(a) $0 < \lim\limits_{n\to\infty} \inf \alpha_n \leq \lim\limits_{n\to\infty} \sup \alpha_n < 1$ and

(b) $0 < \lim\limits_{n\to\infty} \inf \beta_n \leq \lim\limits_{n\to\infty} \sup \beta_n < 1$.

If for a given $x_1 \in C$, the iterative scheme described by (4.3.3), then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T.

For $\gamma_n = 0$ one can obtain Ishikawa type convergence result which is a generalization of Theorem 4.3.8. Unfortunately, just as in [138], it cannot directly deduce the Mann type convergence theorem for the condition $\lim\limits_{n\to\infty} \inf \beta_n > 0$, similar to the condition $1 - \beta_n < 1 - \epsilon, \epsilon > 0$ in [138]. In the following theorem the restriction $\lim\limits_{n\to\infty} \inf \beta_n > 0$ (or $1 - \beta_n < 1 - \epsilon, \epsilon > 0$) is removed for Ishikawa type iteration to refine the result further and unify the proofs of Ishikawa type as well as Mann type convergence.

Theorem 4.3.12[164]. Let C be a nonempty closed bounded and convex subset of an uniformly convex Banach space X. Let T be a completely continuous asymptotically nonexpansive self mapping of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum\limits_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of real numbers in $[0, 1]$ satisfying

(a) $0 < \liminf\limits_{n\to\infty} \alpha_n \leq \limsup\limits_{n\to\infty} \alpha_n < 1$ and

(b) $\limsup\limits_{n\to\infty} \beta_n < 1$.

If for a given $x_1 \in C$, define iterative scheme by (4.3.4), then $\{x_n\}$ and $\{y_n\}$ converge strongly to a fixed point of T.

For $\beta_n = 0$, Theorem 4.3.12 reduces to the following theorem via Mann type convergence result which is a generalization and refinement of Theorem 4.3.7, Theorem 4.3.2 and Theorem 4.3.4.

Theorem 4.3.13[164]. Let C be a nonempty closed bounded and convex subset of a uniformly convex Banach space X. Let T be a completely continuous asymptotically nonexpansive self mapping of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum\limits_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}$ be a sequence of real numbers in $[0, 1]$ satisfying $0 < \liminf\limits_{n\to\infty} \alpha_n \leq \limsup\limits_{n\to\infty} \alpha_n < 1$. For a given $x_1 \in C$, iterative scheme described by (4.3.5). Then $\{x_n\}$ converges strongly to a fixed point of T.

In 2005, Liu [109] not only extended the results of Goebel and Kirk [61], Rhoades [139] and Schu [144] but also corrected the mistakes in Huang [72]. Liu [109] pointed out that the Theorems 4.3.12 and 4.3.13 have no meaning as T is a self mapping of C, $\{x_n\}$ as well as $\{y_n\}$ need not belong to C. Hence $T^n x_n$ and $T^n y_n$ need not be defined.

Theorem 4.3.14[109]. Let C be a nonempty bounded closed and convex subset of a real uniformly convex Banach space X and $T : C \to C$ be a semi-compact asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$. Suppose that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ be four sequences in $[0, 1]$ satisfying the following conditions:

(a) $\alpha_n + \gamma_n \leq 1$, $\beta_n + \delta_n \leq 1$, for all $n \geq n_0$.

(b) there exist positive integers n_0, n_1 and $\epsilon > 0$, $0 < b < \min\{1, \frac{1}{L}\}$ (where $L = \sup\limits_{n\geq 0} k_n$) such that

$$\begin{cases} 0 < \epsilon \leq \alpha_n \leq 1 - \epsilon & \forall\, n \geq n_0 \\ 0 \leq \beta_n \leq b, & \forall\, n \geq n_1, \end{cases} \tag{4.3.6}$$

(c) $\sum\limits_{n=0}^{\infty} \gamma_n < \infty, \sum\limits_{n=0}^{\infty} \delta_n < \infty.$

Then the Ishikawa iterative sequence with errors $\{x_n\}$ defined by (for $x_1 \in C$)

$$\begin{cases} x_{n+1} = (1 - \alpha_n - \gamma_n)x_n + \alpha_n T^n y_n + \gamma_n \omega_n, & n \geq 0 \\ y_n = (1 - \beta_n - \delta_n)x_n + \beta_n T^n x_n + \delta_n z_n, \end{cases} \tag{4.3.7}$$

converges strongly to some fixed point of T in C, where $\{\omega_n\}$ and $\{z_n\}$ are two sequences in C.

The following three lemmas (see Liu [109]) play an important role in the proof of the Theorem 4.3.14.

Lemma 4.3.1. Let X be a real Banach space and $J : X \to 2^{X^*}$ be the normalized duality mapping.Then

$$\|x + y\|^2 \leq \|x\|^2 + \langle y, j(x + y)\rangle$$

for any $x, y \in X$ and for any $j(x + y) \in J(x + y)$.

Lemma 4.3.2. Let $p > 1$ and $r > 0$ be two fixed real numbers. Then a Banach space X is uniformly convex if and only if there exists a continuous strictly increasing convex function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ such that

$$\|\lambda x + (1 - \lambda)y\|^p \leq \lambda\|x\|^p + (1 - \lambda)\|y\|^p - \omega_p(\lambda)f(\|x - y\|)$$

for all $x, y \in B(0, r)$ and $0 \leq \lambda \leq 1$, where $B(0, r)$ is the closed ball of X with center zero and radius r and

$$\omega_p(\lambda) = \lambda^p(1 - \lambda) + \lambda(1 - \lambda)^p. \tag{4.3.8}$$

Lemma 4.3.3. Let C a nonempty closed bounded and convex subset of a real Banach space X and $T : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$, $\lim\limits_{n\to\infty} k_n = 1$. Let $\{x_n\}$ be the Ishikawa iterative sequence with errors described by (4.3.7), in which $\{\gamma_n\}$ and $\{\delta_n\}$ are two sequences in $[0, 1]$ satisfying the following conditions:

$$\sum_{n=0}^{\infty} \gamma_n < \infty \ , \ \sum_{n=0}^{\infty} \delta_n < \infty. \tag{4.3.9}$$

Then $\|x_n - T^n x_n\| \to 0$ implies $\|x_n - Tx_n\| \to 0$.

Proof of Theorem.

From Theorem 4.2.1, T has a fixed point in C. Hence $F(T)$ is nonempty. Rewrite the sequence $\{x_n\}$ defined in (4.3.7) as follows for $x_1 \in C$

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n + u_n, & n \geq 0 \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n + v_n, \end{cases} \tag{4.3.10}$$

where $u_n = \gamma_n(\omega_n - x_n)$ and $v_n = \delta_n(z_n - x_n)$ for all $n \geq 0$. Since C is bounded, ω_n, x_n, z_n all are bounded and thus $\{\omega_n - x_n\}$ and $\{z_n - x_n\}$ both are bounded sequences in X. From (4.3.9) it follows that

$$\sum_n \|u_n\| < \infty, \quad \sum_n \|v_n\| < \infty.$$

i.e.; $\{u_n\}$ and $\{v_n\}$ both are bounded sequences in X.

Take $q \in F(T)$. Since x_n, y_n, $T^n x_n$,$T^n y_n$ all are in C, which implies that there exists an $r > 0$ such that

$$\begin{aligned} C \cup \{x_n - q\} \cup \{y_n - q\} \cup \{x_n - q + u_n\} \cup \{T^n y_n - q + u_n\} \\ \cup \{T^n x_n - q + v_n\} \cup \{x_n - q + v_n\} \subset B(0,r), \end{aligned}$$

where $B(0,r)$ is a closed ball of X with center zero and radius r.

Taking $p = 2$ in Lemma 4.3.2 and $\lambda = \alpha_n$ and (4.3.10) one gets

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|(1-\alpha_n)(x_n - q + u_n) + \alpha_n(T^n y_n - q + u_n)\|^2 \\ &\leq (1-\alpha_n)\|x_n - q + u_n\|^2 + \alpha_n\|T^n y_n - q + u_n\|^2 \\ &\quad - \omega_2(\alpha_n) f(\|x_n - T^n y_n\|). \end{aligned} \tag{4.3.11}$$

Since $\{x_n - q + u_n\}$ and $\{T^n y_n - q + u_n\}$ both are contained in $B(0,r)$, by Lemma 4.3.1 we have

$$\begin{aligned} \|x_n - q + u_n\|^2 &\leq \|x_n - q\|^2 + 2\langle u_n, J(x_n - q + u_n)\rangle \\ &\leq \|x_n - q\|^2 + 2\|u_n\|.\|x_n - q + u_n\| \\ &\leq \|x_n - q\|^2 + 2r\|u_n\|. \end{aligned} \tag{4.3.12}$$

Similarly,

$$\|T^n y_n - q + u_n\|^2 \leq \|T^n y_n - q\|^2 + 2r\|u_n\|. \tag{4.3.13}$$

From (4.3.8), it follows that

$$\omega_2(\alpha_n) = \alpha_n^2(1-\alpha_n) + \alpha_n(1-\alpha_n)^2 = \alpha_n(1-\alpha_n)$$

Substituting the above expression into (4.3.11) and simplifying

$$\begin{aligned}\|x_{n+1}-q\|^2 &\leq (1-\alpha_n)\|x_n-q\|^2+\alpha_n\|T^n y_n-q\|^2+2r\|u_n\|-\alpha_n(1-\alpha_n)f(\|x_n-T^n y_n\|)\\ &= \|x_n-q\|^2+\alpha_n\{\|T^n y_n-q\|^2-\|y_n-q\|^2\}+\alpha_n\{\|y_n-q\|^2-\|x_n-q\|^2\}\\ &\quad+2r\|u_n\|-\alpha_n(1-\alpha_n)f(\|x_n-T^n y_n\|). \qquad (4.3.14)\end{aligned}$$

Consider the third term on the right side of (4.3.14) and using Lemma 4.3.2 with $p=2$, one gets

$$\begin{aligned}\|y_n-q\|^2-\|x_n-q\|^2 &= (1-\beta_n)(x_n-q+v_n)+\beta_n(T^n x_n-q+v_n)\|^2-\|x_n-q\|^2\\ &\leq (1-\beta_n)\|x_n-q+v_n\|^2+\beta_n\|T^n x_n-q+v_n\|^2\\ &\quad-\omega_2(\beta_n)f(\|x_n-T^n x_n\|)-\|x_n-q\|^2\\ &\leq (1-\beta_n)\|x_n-q+v_n\|^2+\beta_n\|T^n x_n-q+v_n\|^2-\|x_n-q\|^2. \qquad (4.3.15)\end{aligned}$$

Since $x_n-q+v_n,\ T^n x_n-q+v_n\in B(0,r)$, it follows from Lemma 4.3.1 that

$$\begin{aligned}\|x_n-q+v_n\|^2 &\leq \|x_n-q\|^2+2\langle v_n, J(x_n-q+v_n)\rangle\\ &\leq \|x_n-q\|^2+2r\|v_n\|. \qquad (4.3.16)\end{aligned}$$

Similarly, we have

$$\|T^n x_n-q+v_n\|^2 \leq \|T^n x_n-q\|^2+2r\|v_n\|. \qquad (4.3.17)$$

Substituting (4.3.16) and (4.3.17) into (4.3.15) and simplifying, we have

$$\begin{aligned}\|y_n-q\|^2-\|x_n-q\|^2 &\leq \beta_n\{\|T^n x_n-q\|^2-\|x_n-q\|^2\}+2r\|v_n\|\\ &\leq \beta_n(k_n^2-1)\|x_n-q\|^2+2r\|v_n\|. \qquad (4.3.18)\end{aligned}$$

Substituting (4.3.18) into (4.3.14) and simplifying, one have

$$\begin{aligned}\|x_{n+1}-q\|^2 &\leq \|x_n-q\|^2+\alpha(k_n^2-1)\|y_n-q\|^2+\alpha_n\{\beta_n(k_n^2-1)\|x_n-q\|^2)+2r\|v_n\|\}\\ &\quad+2r\|u_n\|-\alpha_n(1-\alpha_n)f(\|x_n-T_n^y\|).\\ &\leq \|x_n-q\|^2+\alpha_n(k_n^2-1)\{\|y_n-q\|^2\}+\|x_n-q\|^2\}\\ &\quad+2r(\|u_n\|+\|v_n\|)-\alpha_n(1-\alpha_n)f(\|x_n-T^n y_n\|).\end{aligned}$$

Since $\{x_n-q\}$, $\{y_n-q\}\in B(0,r) \implies \|x_n-q\|\leq r$ and $\|y_n-q\|\leq r$. Besides by condition (4.3.6), $0<\epsilon\leq\alpha_n$ and $\epsilon\leq 1-\alpha_n$ for all $n\geq n_0$. Hence we have

$$\|x_{n+1}-q\|^2 \leq \|x_n-q\|^2+2\alpha_n(k_n^2-1)r^2+2r(\|u_n\|+\|v_n\|)-\epsilon^2 f(\|x_n-T^n y_n\|),\ \forall\, n\geq n_0. \qquad (4.3.19)$$

Set $\sigma=\inf_{n\geq 0}\|x_n-T^n y_n\|$. Next we prove that $\sigma=0$. Let on contrary that $\sigma>0$. Then $\|x_n-T^n y_n\|\geq\sigma>0$ for each $n\geq 0$. Since f is strictly increasing with $f(0)=0$,

$$f(\|x_n-T^n y_n\|)\geq f(\sigma)>0 \text{ for all } n\geq 0. \qquad (4.3.20)$$

It follows from (4.3.19) that for all $n \geq n_0$

$$\|x_{n+1}-q\|^2 \leq \|x_n-q\|^2-\frac{\epsilon^2}{2}f(\sigma)+2\alpha_n(k_n^2-1)r^2+2r(\|u_n\|+\|v_n\|)-\frac{\epsilon^2}{2}f(\sigma). \quad (4.3.21)$$

Since $k_n \to 1$ and $\|u_n\| + \|v_n\| \to 0 (n \to \infty)$, there exists positive integer $n_2 \geq n_0$ such that

$$2\alpha_n(k_n^2 - 1)r^2 + 2r(\|u_n\| + \|v_n\|) < \frac{\epsilon^2}{2}f(\sigma), \ \forall \ n \geq n_2.$$

Hence from (4.3.21), we have

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - \frac{\epsilon^2}{2}f(\sigma), \ \forall \ n \geq n_2,$$

i.e.,
$$\frac{\epsilon^2}{2}f(\sigma) \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2, \ \ \forall \ n \geq n_2.$$

Let $m \geq n_2$ be any positive integer. Then we have

$$\sum_{n=n_2}^{m} \frac{\epsilon^2}{2}f(\sigma) \leq \|x_{n_2} - q\|^2 - \|x_{n+1} - q\|^2 \leq \|x_{n_2} - q\|^2.$$

Letting $m \to \infty$, we get

$$\infty \leq \|x_{n_2} - q\|^2 < \infty,$$

a contradiction. Therefore $\sigma = 0$. By the definition of σ, there exists a subsequence $\{n_j\} \subset \{n\}$ such that

$$\|x_{n_j} - T^{n_j}y_{n_j}\| \to 0 \ \ (n_j \to \infty). \quad (4.3.22)$$

From (4.3.10), we have

$$\begin{aligned}\|x_n - y_n\| &= \|\beta_n(x_n - T^n x_n) - v_n\| \\ &\leq \beta_n\{\|x_n - T^n y_n\| + \|T^n y_n - T^n x_n\|\} + \|v_n\| \quad (4.3.23) \\ &\leq \beta_n\{\|x_n - T^n y_n\| + L\|y_n - x_n\|\} + \|v_n\|\end{aligned}$$

In (4.3.23) taking $n = n_j$ and simplifying, we have

$$\begin{aligned}(1 - L\beta_{n_j})\|x_{n_j} - y_{n_j}\| &\leq \beta\|x_{n_j} - T^{n_j}y_{n_j}\| + \|v_{n_j}\| \\ &\leq \|x_{n_j} - T^{n_j}y_{n_j}\| + \|v_{n_j}\|.\end{aligned}$$

By condition (b), we have $1 - L.\beta_n > 0$ for all $n \geq n_1$ Therefore from (4.3.22), we have

$$\lim_{n_j \to \infty} \|x_{n_j} - y_{n_j}\| \to 0 \quad (4.3.24)$$

It follows from (4.3.24) and (4.3.22) that

$$\|T^{n_j}x_{n_j} - x_{n_j}\| \leq \|T^{n_j}x_{n_j} - T^{n_j}y_{n_j}\| + \|T^{n_j}y_{n_j} - x_{n_j}\|$$

$$\leq L\|x_{n_j} - y_{n_j}\| + \|T^{n_j}y_{n_j} - x_{n_j}\| \tag{4.3.25}$$
$$\to 0 \text{ as } n_j \to \infty.$$

By Lemma 4.3.3, we know

$$\|Tx_{n_j} - x_{n_j}\| \to 0 \text{ as } n_i \to \infty. \tag{4.3.26}$$

Since T is semi compact, there exists a subsequence $\{x_{n_i}\} \subset \{x_{n_j}\}$ such that

$$x_{n_i} \to x* \in C \text{ as } n_i \to \infty. \tag{4.3.27}$$

By the continuity of T, it follows from (4.3.27) that

$$\lim_{n_i\to\infty} \|x_{n_i} - Tx_{n_i}\| = \|x^* - Tx^*\| = 0.$$

i.e., x^* is a fixed point of T in C. Again from (4.3.25), we have

$$\|T^{n_i}x_{n_i} - x^*\| \leq \|T^{n_i}x_{n_i} - x_{n_i}\| + \|x_{n_i} - x^*\| \to 0 \text{ as } n_i \to \infty.$$

Hence from (4.3.10) and (4.3.25), we have

$$y_{n_i} = x_{n_i} - \beta_{n_i}(x_{n_i} - T^{n_i}x_{n_i}) + v_{n_i} \to x^* \text{ as } n_i \to \infty.$$

Again since

$$\|T^{n_i}y_{n_i} - x^*\| \leq L\|y_{n_i} - x^*\|$$

we have

$$T^{n_i}y_{n_i} \to x^* \text{ as } n_i \to \infty. \tag{4.3.28}$$

Now in (4.3.19) taking $q = x^*$, we have

$$\|x_{n_{i+1}} - x^*\|^2 \leq \|x_{n_i} - x^*\|^2 + 2\alpha_n(k_n^2 - 1)r^2 + 2r(\|u_{n_i}\| + \|v_{n_i}\|) - \epsilon^2 f(\|x_{n_i} - T^{n_i}y_{n_i}\|),\ \forall\ n_i \geq 0$$

It follows from (4.3.22), (4.3.27) and the continuity of f that

$$\lim_{n_i\to\infty} \|x_{n_i+1} - x^*\|^2 = 0.$$

$$\text{i.e.,} \qquad x_{n_i+1} \to x^* \text{ as } n_i \to \infty. \tag{4.3.29}$$

Therefore, we have

$$\|T^{n_i+1}x_{n_{i+1}} - x^*\| \leq L\|x_{n_i+1} - x^*\| \to 0 \text{ as } n_i \to \infty \tag{4.3.30}$$

By (4.3.10), (4.3.29) and (4.3.30), one gets

$$y_{n_i+1} = x_{n_i+1} - \beta_{n_i+1}(T^{n_i+1}x_{n_i+1} - x_{n_i+1}) + v_{n_i+1} \to x^* \text{ as } n_i \to \infty \tag{4.3.31}$$

Therefore we have

$$\|T^{n_i+1}y_{n_i+1} - x^*\| \leq L\|y_{n_i+1} - x^*\| \to 0 \text{ as } n_i \to \infty \tag{4.3.32}$$

Continuing in this way, by induction we can prove that for any $m \geq 0$,

$$x_{n_i+m} \to x^*, \quad y_{n_i+m} \to x^* \quad (n_i \to \infty),$$

$$T^{n_i+m} x_{n_i+m} \to x^*, \quad T^{n_i+m} y_{n_i+m} \to x^* \quad (n_i \to \infty).$$

Next we prove that $x_n \to x^*$ $(n \to \infty)$. In fact, by the definition of superior limit and inferior limit we have

$$\begin{aligned} 0 &\leq \liminf_{n\to\infty} \|x_n - x^*\| \leq \limsup_{n\to\infty} \|x_n - x^*\| \\ &\leq \sup_m \{\|x_{n_i+m} - x^*\|\} \text{ for any } n_i. \end{aligned}$$

Denote $\alpha_i = \sup_m \|x_{n_i+m} - x^*\|$. Then the sequence $\{\alpha_i\}$ is non increasing. Therefore we have

$$\begin{aligned} 0 &\leq \liminf_{n\to\infty} \|x_n - x^*\| \leq \limsup_{n\to\infty} \|x_n - x^*\| \\ &\leq \lim_{n_i\to\infty} \sup_m \|x_{n_i+m} - x^*\| \\ &= \lim_{n_i\to\infty} \sup_m \|x_{n_i+m} - x^*\| \\ &= \sup_m \lim_{n_i\to\infty} \|x_{n_i+m} - x^*\| \\ &= 0 \end{aligned}$$

i.e., $x_n \to x^*$. This completes the proof.

Theorem 4.3.15. Let C be a nonempty closed bounded and convex subset of a uniformly convex real Banach space X and $T : C \to C$ be a semi compact asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty), k_n \to 1$. Suppose that $\{\alpha_n\}$ and $\{\gamma_n\}$ be two sequences in $[0, 1]$ satisfying the following conditions:

(a) $\alpha_n + \gamma_n \leq 1, \forall\, n \geq 0$;

(b) there exist positive integers n_0 and $\epsilon > 0$ such that

$$0 < \epsilon \leq \alpha_n \leq 1 - \epsilon, \ \forall\, n \geq n_0.$$

(c) $\sum_{n=0}^{\infty} \gamma_n < \infty$.

Then the Mann iterative sequence with errors $\{x_n\}$ defined by

$$\begin{cases} x_0 \in C \\ x_{n+1} = (1 - \alpha_n - \gamma_n)x_n + \alpha_n T^n x_n + \gamma_n \omega_n, & n \geq 0 \end{cases}$$

converges strongly to some fixed point of T where $\{\omega_n\}$ is a sequence in C.

Proof. This theorem can be proved by taking $\beta_n = 0$ and $\delta_n = 0$ for all $n \geq 0$, in Theorem 4.3.14.

§ 4.4. Contractive mappings

Definition 4.4.1[125]. A self mapping T defined on a subset C of a Banach space X is said to be *asymptotically contractive on* C if there exists $x_0 \in C$ such that

$$\lim_{\|x\| \to \infty} \sup_{x \in C} \frac{\|T(x) - T(x_0)\|}{\|x - x_0\|} < 1. \tag{4.4.1}$$

This condition is independent of the choice of $x_0 \in C$.

Luc [110, 2002] extended Theorem 2.2.3 for an unbounded set using the idea of asymptotically compact set.

Definition 4.4.2[125]. A subset C of a Banach space X is said to be *asymptotically compact* if for any sequence $\{x_n\}$ of C such that $\{r_n\} = \{\|x_n\|\} \to \infty$, the sequence $\{r_n^{-1} x_n\}$ has a convergent subsequence.

Theorem 4.4.1[110]. Let C be a nonempty closed convex and asymptotically compact set in X. Let T be a nonexpansive and asymptotically contractive mapping from C to itself. Then $F(T)$ is nonempty closed and convex.

In 2003, Penot [125] presented a fixed point theorem for a nonexpansive mapping defined on a closed convex subset of a uniformly convex Banach space into itself under some asymptotic contraction assumptions. Penot's result generalizes Theorem 2.2.3 due to Browder-Göhde-Kirk and also extends Theorem 4.4.1 due to Luc [110] by dropping the condition of compactness.

Theorem 4.4.2[125]. Let C be a closed convex subset of a reflexive Banach space X. If $T : C \to X$ be a nonexpansive mapping which is asymptotically contractive on C such that $T(C) \subset C$ and $I - T$ is demiclosed then T has a fixed point.

Corollary 4.4.1. Let C be a closed convex subset of a uniformly convex Banach space X. Let $T : C \to X$ be a nonexpansive asymptotically contractive mapping on C such that $T(C) \subset C$. Then T has a fixed point.

Suzuki [155] proved fixed point theorem for nonexpansive mappings whose domains are unbounded subsets of Banach spaces. The following theorem generalizes the Theorem 4.4.2 due to Penot [125].

Theorem 4.4.3[155]. Let C be an unbounded closed convex subset of a uniformly convex Banach space X. Let T be a nonexpansive mapping on C. Suppose that T

is asymptotically contractive. Then T has a fixed point.

In 1967, Browder [22] introduced the notion of pseudocontractive mappings and proved the next theorem.

Definition 4.4.3. Let T be a self mapping on a Banach space X, then T is said to be *pseudocontractive* if for all $x, y \in X$ and for all $k > 0$,

$$\|x - y\| \leq \|(1 + k)(x - y) - k(Tx - Ty)\|.$$

Clearly if T is nonexpansive, then T is pseudocontractive since

$$\|(1 + k)(x - y) - k(T(x) - T(y))\| \geq (1 + k)\|x - y\| - k\|T(x) - T(y)\|.$$

Theorem 4.4.4. Let B be a closed ball in a uniformally convex Banach space X and C an open set containing B. Let T be a pseudocontractive mapping of C into X such that T maps the boundary of B into B. Suppose also that T is demicontinuous and that
(a) T is uniformly continuous in the strong topology on bounded subsets of X, or
(b) X^* is uniformly convex.
Then T has a fixed point in B.

Kirk [88, 1970] modified Theorem 4.4.4 as follows:

Theorem 4.4.5[88]. Let X be a uniformly convex Banach space and B a closed ball in X. Let T be a Lipschitzian pseudocontractive mapping of B into X such that T maps boundary of B into B. Then T has a fixed point in B.

In 1972, Assad and Kirk [3] modified the approach of Kirk [88] and demonstrated how fixed point theorems for pseudocontractive mappings may be derived from the fixed point theorems of nonexpansive mappings.

Theorem 4.4.6[3]. Let C be a closed convex subset of a reflexive Banach space X and K a nonempty closed bounded and convex subset of C which possesses normal structure. Let T be a Lipschitzian pseudocontractive mapping of K into C such that $T(x) \in K$ when $x \in \delta_C K$. Then T has a fixed point.

The connection between pseudocontractive mappings have been further refined by Bruck [31] has made an interesting observation that if C is a closed convex set which has the f.p.p. for nonexpansive mappings and if $T : C \to C$ is a Lipschitzian local pseudocontraction, then T always has a fixed point.

In 1991, Schu [144] introduced a class of asymptotically pseudocontractive mapping in Hilbert spaces and proved strong convergence theorems using modified Ishikawa iteration scheme.

Definition 4.4.4[144]. Let C be a nonempty subset of a normed space $X, T : C \to C$; $\{k_n\} \subset [1, \infty)$. T is said to be *asymptotically pseudocontractive with sequence* $\{k_n\} \iff \lim k_n = 1$ for all $n \in N$ and all $x, y \in C$ there is $j \in J_X(x - y)$ such that $j(T^n(x) - T^n(y)) \leq k_n \|x - y\|^2$ where J_X is the normalized duality mapping.

Remark 4.4.1.

(*a*) If T is asymptotically nonexpansive, then for all $j \in J_X(x - y)$,

$$j(T^n x - T^n y) \leq \|x - y\| \|T^n x - T^n y\| \leq k_n \|x - y\|^2.$$

Hence every asymptotically nonexpansive mapping is asymptotically pseudocontractive too and it also remains uniformly k-Lipschitzian for some $k > 0$.

(*b*) Rhoades [137] showed that the class of asymptotically nonexpansive mappings is a proper subclass of the class of asymptotically pseudocontractive mappings. For $x \in [0, 1]$, define $T(x) = (1 - x^{2/3})^{3/2} \in [0, 1]$. Then T is not Lipschitzian and so it can't be asymptotically nonexpansive. But since $T \circ T = I$ and T is monotonically decreasing, it follows that

$$(x - y)(T^n x - T^n y) = |x - y|^2 \quad \forall\, n \in 2N$$

and

$$\begin{aligned}(x - y)(T^n x - T^n y) &= (x - y)(Tx - Ty) \leq 0 \\ &\leq |x - y|^2 \;\forall\, n \in 2N - 1.\end{aligned}$$

Hence T is asymptotically pseudocontractive with constant sequence $\{1\}_{n \in N}$.

Theorem 4.4.7[144]. Let C be a nonempty closed bounded and convex subset of a Hilbert space X. For $k > 0$; $T : C \to C$ be completely continuous, uniformly k-Lipschitzian and asymptotically pseudocontractive mapping with sequence $\{k_n\} \in [1, \infty)$; $q_n = 2k_n - 1$ for all $n \in N$; $\sum\limits_{n=1}^{\infty} ({q_n}^2 - 1) < \infty$; $\{\alpha_n\}, \{\beta_n\} \in [0, 1]^N$; $\epsilon \leq \alpha_n \leq \beta_n \leq b$ for all $n \in N$, some $\epsilon > 0$ and some $b \in (0, k^{-2}[(1 + k^2)^{\frac{1}{2}} - 1])$; $x_1 \in C$; for all $n \in N$, define

$$z_n = (1 - \beta_n) x_n + \beta_n T^n x_n \;\text{ and }\; x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T^n z_n.$$

Then $\{x_n\}$ converges strongly to some fixed point of T.

Remark 4.4.2. It would be interesting to know whether a continuous asymptotically pseudocontractive self mapping of a closed bounded and convex subset of a uniformly convex Banach space always possesses a fixed point. This is indeed the case for continuous pseudocontractive mapping.

Sharma and Sahu [151, 2000] dropped the compactness condition from Theorem 4.4.7 and gave the existence of the fixed point of asymptotically pseudocontractive mappings in Banach spaces. Before stating the theorem we give the following results which will be used in the proof the next theorem.

Theorem 4.4.8[151]. Let C be a nonempty closed convex subset of a Banach space X. Let $T : C \to C$ is asymptotically pseudocontractive with sequences $\{k_n\}$ and $\{\alpha_n\} \in (0,1)$ for all $n \in N$ with $\lim \alpha_n = 1$. Then

(a) for each $n \in N$ there is exactly one $x_n \in C$ such that

$$x_n = \left(\frac{\alpha_n}{k_n}\right) T^n x_n, \quad \text{where } \alpha_n \in (0,1),$$

(b) if C is bounded and T is uniformly asymptotically regular and uniformly k-Lipschitzian, it follows that $\inf\{\|x - Tx\| : x \in C\} = 0$.

Lemma 4.4.1[151]. Let C be a nonempty closed convex subset of a uniformly convex Banach space X possessing a weakly sequential continuous duality mapping. Let $T : C \to C$ be a uniformly k-Lipschitzian and asymptotically pseudocontractive mapping with sequence $\{k_n\}$ satisfying the condition

$$\|x - T^n y\|^2 \leq j(x - T^n y), \ \forall \ x, y \in C, n \in N,$$

where $j \in J_X(x - y)$. Then $(I - T)$ is demiclosed with respect to 0.

Theorem 4.4.9[151]. Let C be a nonempty closed bounded and convex subset of a uniformly convex Banach space X possessing a weakly sequentially continuous duality mapping. Suppose $T : C \to C$ is uniformly asymptotically regular, uniformly k- Lipschitzian and asymptotically pseudocontractive mapping with sequence $\{k_n\}$ and satisfy the condition

$$\|x - T^n y\|^2 \leq j(x - T^n y), \forall \ x, y \in C, n \in N,$$

where $j \in J_X(x - y)$. Then $F(T) \neq \emptyset$.

Proof.

By Theorem 4.4.8, for each $n \in N$ there exists one $x_n \in C$ such that $x_n = (\alpha_n/k_n)T^n x_n$, where $\alpha_n \in (0,1)$ with $\lim \alpha_n = 1$, since $\{x_n\}$ is bounded. Therefore, there is a constant q such that $\|x_n - T^n x_n\| = q|1 - k_n/\alpha_n|$ for all $n \in N$ and so $\|x_n - T^n x_n\| \to 0$ as $n \to \infty$. By the asymptotically regularity of T, we have

$$\begin{aligned}
\|x_n - Tx_n\| &\leq \|x_n - T^n x_n\| + \|T^n x_n - Tx_n\| \\
&\leq \|x_n - T^n x_n\| + k\|T^{n-1}x_n - x_n\| \\
&\leq \|x_n - T^n x_n\| + k(\|T^{n-1}x_n - T^n x_n\| + \|T^n x_n - x_n\|) \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}$$

Since X is reflexive and $\{x_n\}$ is bounded, there exists $z \in X$ and a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to z$ (weakly).

Furthermore, since $x_{n_i} - Tx_{n_i} \to 0$ and by Lemma 4.4.1, $(I - T)$ is demiclosed at zero, it follows that $z = Tz$. This completes the proof.

Definition 4.4.5. A normed space X has property (U, b, α, β) if and only if

$$\|x + y\|^\alpha + b\|x - y\|^\alpha - 2^\beta(\|x\|^\alpha + \|y\|^\alpha) \geq 0, \quad \forall\, x, y \in X.$$

Theorem 4.4.10[151]. Let C be a nonempty closed convex bounded subset of a Banach space X with the property $(U, \lambda, m+1, m), \lambda \in \Re, m \in N$. Let $T : C \to C$ is uniformly k-Lipschitzian for some $k > 0$ and asymptotically pseudocontractive mapping with sequence $\{k_n\}$ and also T satisfies condition B(see, Def. 2.3.2). Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers satisfying the following conditions:

(a) $0 < a \leq \alpha_n \leq \alpha < 1$ and $0 < b \leq \beta_n \leq \beta < 1$, $\forall\, n \in N$,

(b) $\sum\limits_{n=1}^{\infty}(v_n c - m) < \infty$, where $v_n = (m+1)k_n - m$ and $c = \{\frac{\lambda}{2^m - 1}\}$,

(c) $(1 - 2\beta^m c - \beta^{m+1}k^{m+1}c)c + 1 - \beta^m c - c^2 > 0$ and $1 - \alpha^m c - (1 - mb)c^2 > 0$.

For $x_1 \in C$, define

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n y_n \\ y_n = (1 - \beta_n)x_n + \beta T^n x_n \quad \forall\, n \in N. \end{cases}$$

If $\lim d(x_n, F(T))$ exists and $F(T)$ is closed, then the sequence $\{x_n\}$ converges strongly to an element of $F(T)$.

Remark 4.4.3. Theorem 4.4.10 improves Theorem 4.4.7.

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